4. INTEGRATION IN VECTOR FIELDS
September 16, 2009

1 Line integrals

Consider a wire W in either the plane or space, with variable mass density. It is natural to ask for the total mass of W. In fact, a special case of this is usually discussed in first year calculus where the wire in question is assumed to be a straight line segment, say the interval $[a, b]$ on the x-axis. The wire's mass is then given as

$$\int_a^b \delta(x)dx \tag{1}$$

where $\delta(x)$ is the mass density at $(x, 0)$. The rationale behind this is that if the length of a small segment on the axis is denoted by Δx, then its mass is (approximately) the product

$$\delta(x)\Delta x$$

and hence, by the Fundamental Theorem of Calculus the total mass of the wire is

$$\int_a^b \delta(x)dx.$$

This motivates the following definition. If $f : D \to \Re$ and $\mathbf{C}(t) \subset D, t \in [a, b]$ then the (*scalar*) *line integral of f along* $\mathbf{C}$ is denoted by

$$\int_{\mathbf{C}} f ds \tag{2}$$

where s denotes arclength along $\mathbf{C}$. Since arclength parametrizations are hard to come by, we use the equation

$$\frac{ds}{dt} = |\mathbf{C}'(t)|$$

and the method of integration by substitution to convert the integral of (2) into the easily computed

$$\int_a^b f(\mathbf{C}(t))|\mathbf{C}'(t)|dt \tag{3}$$

Example 1.1 *A wire has the shape of the curve* $\mathbf{C}(t) = (t, t^2, t^3), 0 \le t \le \pi$ *and mass density*

$$\delta(x, y, z) = \sin x.$$

Find the total mass of this wire.

Clearly

$$|\mathbf{C}'(t)| = |(1, 2t, 3t^2)| = \sqrt{1 + 4t^2 + 9t^4}$$

Hence, by Eq'n 3, the total mass is

$$\int_0^\pi \sin t \sqrt{1 + 4t^2 + 9t^4} dt = 479.20 \ \ldots.$$

Equation (3) can be given an alternative, more symbolic, interpretation. Recall that if s denotes arclength along the curve $\mathbf{C}(t)$, then

$$\frac{ds}{dt} = |\mathbf{C}'(t)|$$

or, pretending that ds and dt are genuine numbers,

$$ds = |\mathbf{C}'(t)| dt$$

which explains how (2) can be transformed into(3).

It is tempting to ask for an interpretation of the line integral as an area. After all, much time was spent in first year calculus explaining how areas can be computed using definite integrals and many students identify the two. This cannot be done for line integrals in a natural way . Something of interest will be pointed out below, but the readers should abandon the notion that integrals must represent areas. It is much better for the sequel to think of any integral as a continuous summation process.

The scalar line integral was defined in terms of an arclength parametrization of the underlying curve. Such choices are not unique and could, in principle, affect the value of the integral. We proceed to show that such is not the case. This is not obvious as another important integral will be defined below whose value does indeed depend on the parametrization.

Let s and $\bar{s}$ denote two arclength parameters of $\mathbf{C}$ where

$$a \le s \le b, \quad c \le \bar{s} \le d.$$

Geometric considerations make it clear that there is a constant k such that

$$s = \bar{s} + k \quad \text{or} \quad s = -\bar{s} + k.$$

Define

$$\bar{f}(\bar{s}) = f(s).$$

In the first case,

$$\int_{\mathbf{C}} f ds = \int_a^b f(s)ds = \int_c^d f(\bar{s}+k)\frac{ds}{d\bar{s}}d\bar{s} = \int_c^d f(\bar{s}+k)d\bar{s} = \int_{\mathbf{C}} \bar{f} d\bar{s}.$$

In the second case,

$$\int_{\mathbf{C}} f ds = \int_a^b f(s)ds = \int_d^c f(\bar{s}+k)\frac{ds}{d\bar{s}}d\bar{s} = \int_c^d f(-\bar{s}+k)(-1)d\bar{s} = \int_{\mathbf{C}} \bar{f} d\bar{s}.$$

Proposition 1.2 *The scalar line integral is independent of the arclength parametrization.*

□

A *vector field* is a function $\mathbf{F} : \Re^n \rightarrow \Re^n$.

$$\mathbf{F}(x,y) = (x^2 + y, y^3 - \sin x)$$

$$\mathbf{F}(x,y) = (e^x - 1 + 2y, \ln(1+y^2) - 1)$$

$$\mathbf{F}(x,y,z) = (xyz, xy, x)$$

$$\mathbf{F}(x,y,z,w) = (w, x, y, z)$$

It is a good idea to visualize a vector field as a collection of arrows $\mathbf{F}(x,y)$ each of which is anchored at its domain point (x,y).

Example 1.3 *Use a computer package to draw the vector field*

$$\mathbf{F}(x,y) = (x,y)$$

See Figure 1.

Example 1.4 *Use a computer package to draw the vector field*

$$\mathbf{F}(x,y) = (-y,x)$$

See Figure 2.

Example 1.5 *Use a computer package to draw a 2-dimensional gravitational field.*

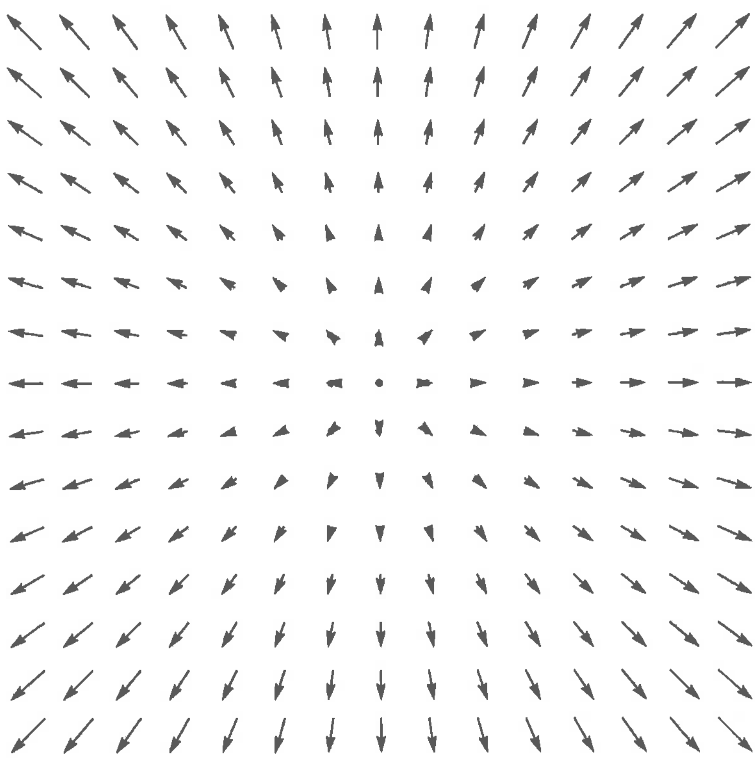


Figure 1: A source-like vector field

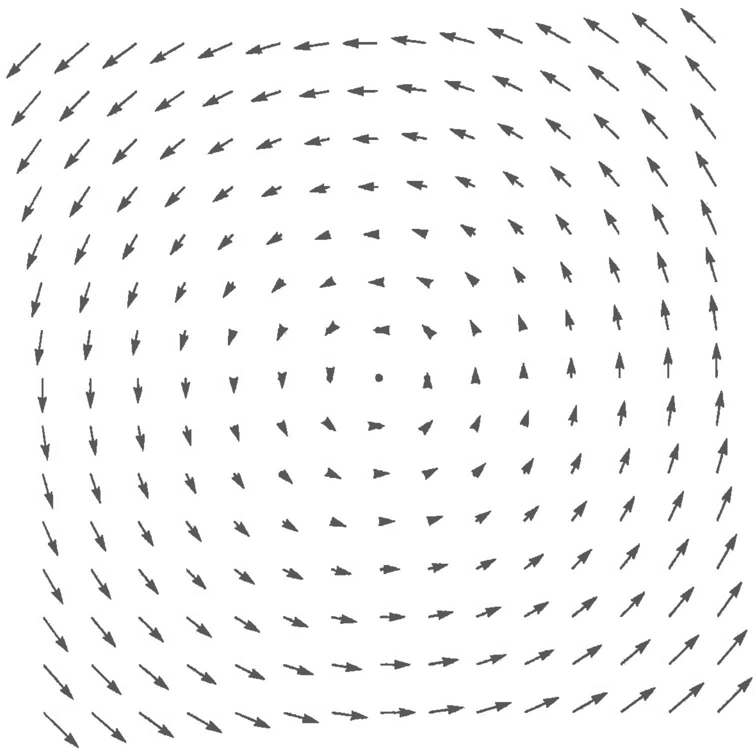


Figure 2: A rotational vector field

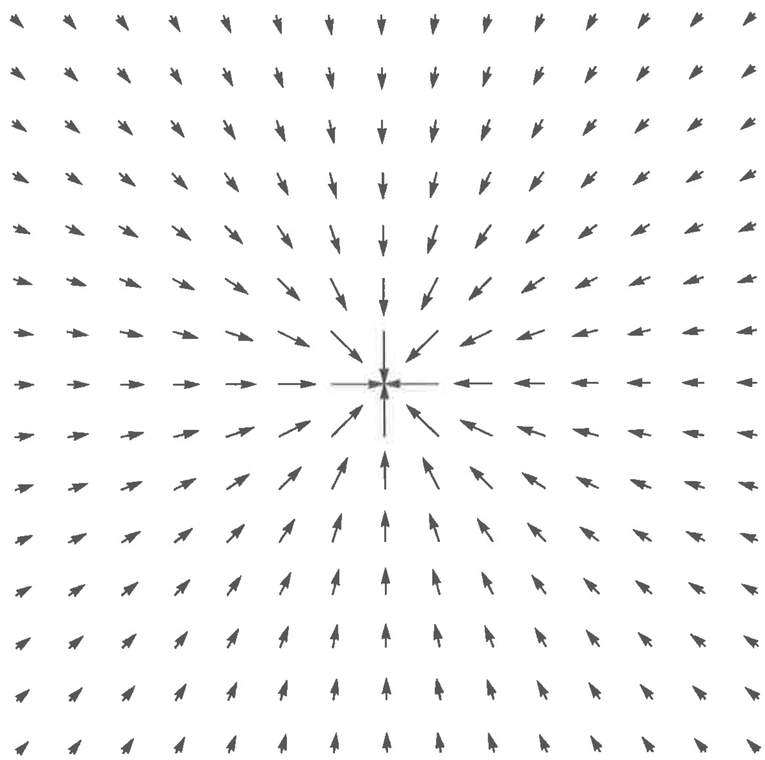


Figure 3: A gravitational vector field

A fixed mass at the origin induces a gravitational force $\mathbf{F}$ on a unit mass at the point $\mathbf{r}$ that is directed towards the origin and whose length is inversely proportional to $|\mathbf{r}|^2$. Since the outward directed unit vector at $\mathbf{r}$ is

$$\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(x,y)}{\sqrt{x^2+y^2}}$$

it follows that, for some positive number c,

$$\mathbf{F}(x,y) = \frac{-(x,y)}{\sqrt{x^2+y^2}} \cdot \frac{c}{(x^2+y^2)} = \frac{-c(x,y)}{(x^2+y^2)^{3/2}}.$$

This field is displayed in Figure 4.

Steady state flows, that is, flows whose velocity vector at each point remain constant over time, also constitute vector fields. For example in Figure 4 is displayed the velocity field of a straight wide river. Figure 5 portrays the vector field about a whirlpool. For such 2-dimensional flows, it is reasonable to ask how much flows across an arc and how much flow along an arc. Higher dimensional flows will be discussed subsequently.

We next stipulate a (velocity) flow

$$\mathbf{F}(x,y) = (F_1(x,y), F_2(x,y)) \quad \text{or} \quad \mathbf{F} = (F_1, F_2)$$

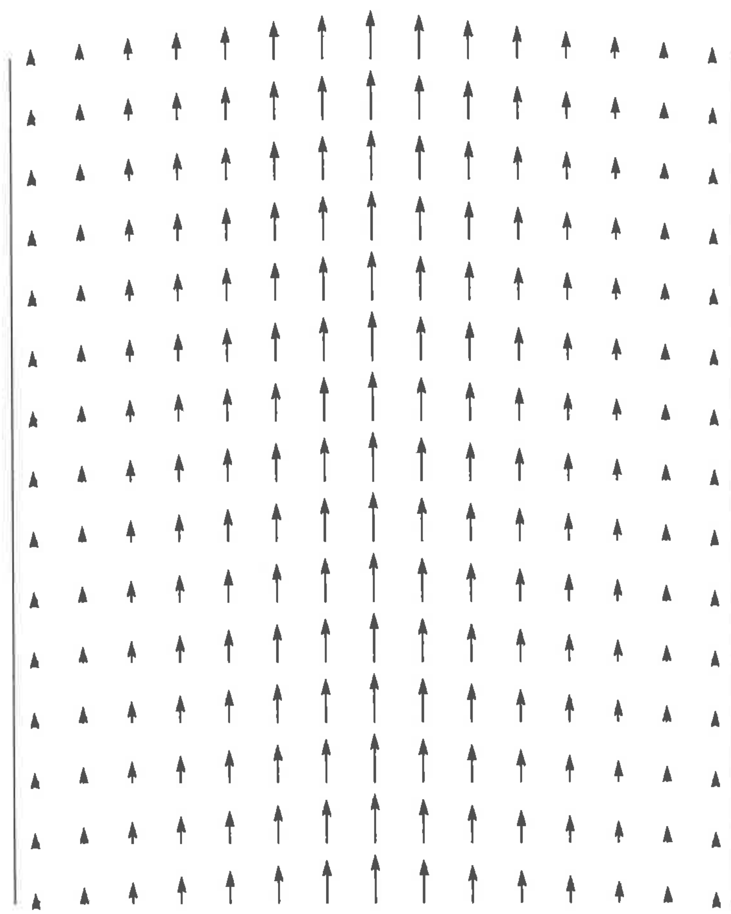


Figure 4: The velocity flow of a straight wide river

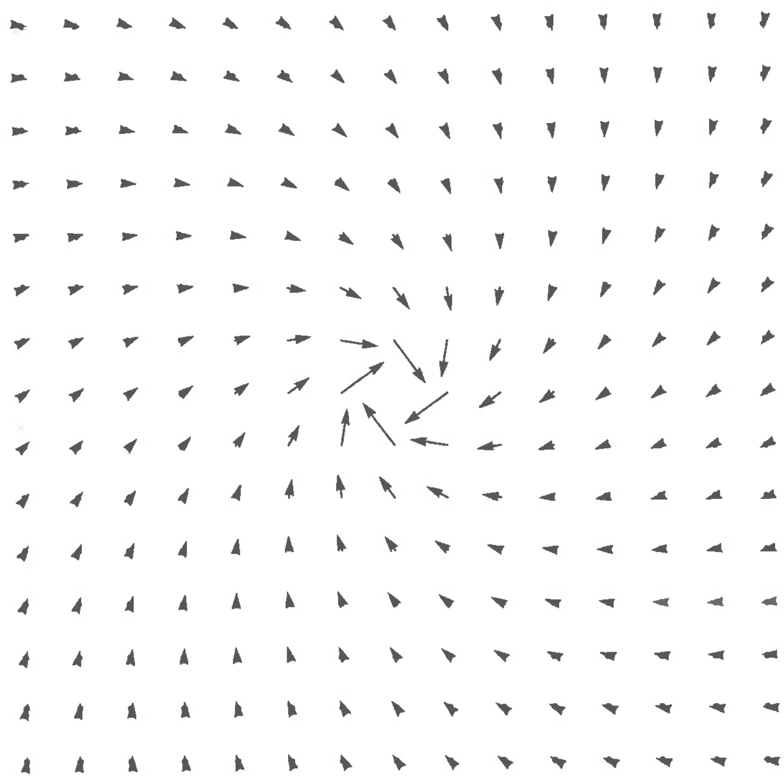


Figure 5: The velocity flow of a whirlpool

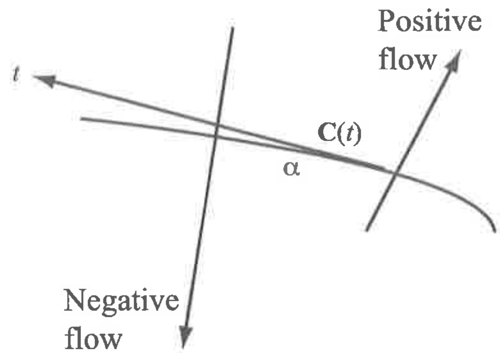


Figure 6: The unit normal and tangent to a curve

and a parametrized curve

$$\mathbf{C}(t) = (x(t), y(t)), \quad a \leq t \leq b.$$

If $\mathbf{t} = \mathbf{t}(t)$ denotes the unit tangent of $\mathbf{C}(t)$, i.e., if

$$\mathbf{t}(t) = \frac{\mathbf{C}'(t)}{|\mathbf{C}'(t)|}$$

and Δs denotes the length of a small arc α of $\mathbf{C}$ containing the point $\mathbf{C}(t)$, then the infinitesimal flow of $\mathbf{F}$ along α equals the length of the projection of $\mathbf{F}$ onto $\mathbf{t}$ multiplied by ds, or, symbolically,

$$\mathbf{F} \cdot \mathbf{t} ds.$$

Consequently, the total flow along $\mathbf{C}$, otherwise known as the *circulation* of $\mathbf{F}$ along $\mathbf{C}$, is (defined as)

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{t} ds.$$

Since, as noted above, arclength parametrizations are hard to come by, we again make use of the equation

$$\frac{ds}{dt} = |\mathbf{C}'(t)|$$

to convert this symbolic scalar line integral into the easily computed form

$$\int_a^b \mathbf{F} \cdot \frac{\mathbf{C}'(t)}{|\mathbf{C}'(t)|} |\mathbf{C}'(t)| dt = \int_a^b \mathbf{F} \cdot \mathbf{C}'(t) dt \tag{4}$$

Example 1.6 *Let* $\mathbf{F} = (xy^2, y + x^3)$ *and* $\mathbf{C}(t) = (3t^2, 2t^3), 1 \leq t \leq 2$.. *Evaluate*

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}.$$

By Eq'n (4)

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} = \int_1^2 ((3t^2)(2t^3)^2, (2t^3 + (3t^2)^3) \cdot (6t, 6t^2)dt \approx 16,689.6.$$

Example 1.7 *Let* $\mathbf{F} = (xyz^2, 2x^2yz, 3xy^2z)$ *and* $\mathbf{C}(t) = (2t, 3t^2, 5t^3), 0 \leq t \leq 2.$. *Evaluate*

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}.$$

By Eq'n (4)

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$$

$$= \int_0^2 ((2t)(3t^2)(5t^3)^2, 2(2t)^2(3t^2)(5t^3), 3(2t)(3t^2)^2(5t^3)) \cdot ((2, 6t, 15t^2))dt$$

$$\approx 825,716.$$

Since $\mathbf{F} = (F_1, F_2)$ and $\mathbf{C}'(t) = (x'(t), y'(t))$ the expression in (4) can be converted to

$$\int_a^b (F_1 x'(t) + F_2 y'(t))dt = \int_a^b F_1 x'(t)dt + \int_a^b F_2 y'(t)dt.$$

The substitutions $x = x(t)$ and $y = y(t)$ convert these integrals into

$$\int_{\mathbf{C}} F_1 dx + \int_{\mathbf{C}} F_2 dy.$$

The sum of these integrals is denoted by the suggestive symbol

$$= \int_{\mathbf{C}} F_1 dx + F_2 dy. \tag{5}$$

Example 1.8 *Evaluate* $\int_C (x+y)dx + xydy$ *where* $\mathbf{C} = (t, t^2), 1 \leq t \leq 2$.

Here $\mathbf{F} = (x + y, y)$. Consequently

$$\int_C (x+y)dx + xydy = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$$

$$= \int_1^2 ((t + t^2), tt^2) \cdot (1, 2t)dt = 487/30.$$

Example 1.9 *Evaluate* $\int_C (x+y+z)dx + xyzdy - xdz$ *where* $\mathbf{C} = (t, t^2, t^3), 0 \leq t \leq 1$.

Here $\mathbf{F} = (x + y + z, xyz, -x)$. Consequently

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} = \int_0^1 (t + t^2 + t^3, tt^2t^3, -t) \cdot (1, 2t, t^2)dt = 35/6.$$

Example 1.10 *Evaluate* $\int_C (x+y+z)dx + xyzdy - xdz$ *where* **C** *is the line segment from the origin to the point (1, 2, 3).*

Here $\mathbf{F} = (x+y+z, xyz, -x)$. The line segment **C** has the parametric equation

$$\mathbf{C}(t) = (t, 2t, 3t)$$

and hence

$$\mathbf{C}'(t) = (1, 2, 3).$$

Hence,

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} = \int_0^1 (t+2t+3t, t(2t)(3t), -t) \cdot (1,2,3)dt$$

$$= \int_0^1 (6 + 12t^3 - 3t)dt = 7.5.$$

Example 1.11 *Evaluate* $\int_C (x+y)dx + xydy$ *where* **C** *is the polygonal line that consists of the horizontal line from (1,1) to (3,1) followed by the vertical line from (3,1) to (3,5).*

Here $\mathbf{F} = (x+y, xy)$. Let $\mathbf{C}_1$ denote the segment from (1,1) and $\mathbf{C}_2$ the segment from (3,1) to (3,5). The segments have the following parametrizations:

$$\mathbf{C}_1(t) = (t, 1), \quad 1 \le t \le 3$$

$$\mathbf{C}_2(t) = (3, t), \quad 1 \le t \le 5.$$

Hence,

$$\int_{\mathbf{C}_1} \mathbf{F} \cdot \mathbf{ds} = \int_1^3 (t+1, t) \cdot (1, 0)dt = 6$$

and

$$\int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{ds} = \int_1^5 (3+t, 3t) \cdot (0, 1)dt = 36.$$

Consequently

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} = \int_{\mathbf{C}_1} \mathbf{F} \cdot \mathbf{ds} + \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{ds} = 6 + 36 = 42.$$

Example 1.12 *Evaluate* $\int_C (x^2+y^2)dx + 2xydy$ *where* **C** *is the counter-clockwise circle of radius 2 centered at the origin.*

Here $\mathbf{F} = (x^2+y^2, 2xy)$ and $\mathbf{C}(t) = 2(\cos t, \sin t), \quad 0 \le t \le 2\pi$. Consequently,

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$$

$$= \int_0^{2\pi} (4\cos^2 t + 4\sin^2 t, 2\cos t \sin t) \cdot (-\sin t, \cos t)dt = 0.$$

Let $\mathbf{F}$ be a flow and $\mathbf{C}$ a curve in $\Re^2$. To quantify the total flow of $\mathbf{F}$ across $\mathbf{C}$, it is necessary first to establish a convention regarding positive and negative flows across a curve. We shall agree to regard as *positive* a flow that crosses the curve from left to right from the point of view of the tangent to $\mathbf{C}$.

The flow across the small arc α is the product of the projection of $\mathbf{F}$ onto the normal $\mathbf{n}(t)$ at $\mathbf{C(t)}$) with the length of α, that is,

$$\mathbf{F} \cdot \mathbf{n} \Delta s.$$

Hence the total flow across $\mathbf{C}$ is

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds.$$

This is the symbolic form of the total flow of $\mathbf{F}$ across $\mathbf{C}$. Its computational form is obtained by recalling that the positive flow direction is obtained by a clockwise 90° rotation of the tangent, the total flow becomes

$$\int_a^b (F_1, F_2) \cdot \frac{(y'(t), -x'(t))}{|\mathbf{C}'(t)|} |\mathbf{C}'(t)| dt$$

$$= \int_a^b [-F_2 x'(t) + F_1 y'(t)] dt \qquad (6)$$

$$= \int_{\mathbf{C}} -F_2 dx + F_1 dy.$$

Example 1.13 *Let* $\mathbf{F} = (xy^2, y + x^3)$ *and* $\mathbf{C}(t) = (3t^2, 2t^3), 1 \leq t \leq 2$. *Evaluate the total flow of* $\mathbf{F}$ *across* $\mathbf{C}$. *By Eq'n (6)*

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds = \int_1^2 -(2t^3 + (3t^2)^3) \cdot 6t + 3t^2 (2t^3)^2 \cdot 6t^2 dt$$

$$= 2922.25.$$

Since $\mathbf{C}'(1) = (6, 6)$ *it follows that the overall direction of the flow is from the side of* $\mathbf{C}$ *containing (0, 1) to the side containing (1, 0).*

Example 1.14 *Let* $\mathbf{F} = (x^2 - y^2, y + x)$. *Find the total flow of* $\mathbf{F}$ *across the line segment from* (0,1) to (1, 0).

The line segment in question can be parametrized as

$$\mathbf{C}(t) = (t, 1-t), \quad 0 \leq t \leq 1.$$

Hence

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds = \int_0^1 [-((1-t)+t)) \cdot 1 + (t^2 - (1-t)^2)(-1)] dt = -1.$$

Since $\mathbf{C}'(t) = (1, -1)$ it follows that the positive direction is from the origin to the point (2, 2). Consequently we have a total flow of 1 in the opposite direction, from (2, 2) to the origin.

Example 1.15 *Let* $\mathbf{F} = (x^2 - y^2, y + x)$. *Find the total flow of* $\mathbf{F}$ *across the polygonal line* $\mathbf{C}$ *from* (1, 1) *to* (1,3) *and from there on to* (5,3).

Note that $dx = 0$ along the first part of $\mathbf{C}$ and $dy = 0$ along the second part. Consequently, the total flow across $\mathbf{C}$ is

$$\int_{\mathbf{C}} -(x+y)dx + (x^2 - y^2)dy$$

$$= 0 + \int_1^3 (1 - y^2)dy + \int_1^5 -(x+3)dx + 0 = -\frac{92}{3}.$$

Example 1.16 *Determine the total flow of the field*

$$\mathbf{F} = \frac{-(x,y)}{(x^2+y^2)^{3/2}}$$

across the circle $\mathbf{C}$ *of centered at the origin.*

The circle $\mathbf{C}$ can be parametrized as $2(\cos t, \sin t), 0 \leq t \leq 2\pi$. Consequently the total flow across $\mathbf{C}$ is

$$\int_{\mathbf{C}} \frac{y}{(x^2+y^2)^{3/2}} dx - \frac{x}{(x^2+y^2)^{3/2}} dy$$

$$= \int_0^{2\pi} \left[\frac{2\sin t}{8}(-2\sin t) - \frac{2\cos t}{8}(2\cos t) \right] dt = -\pi.$$

Since the parametrization is counterclockwise the unsigned total flow is directed into the circle.

It is necessary to investigate the extent to which line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} \tag{7}$$

Figure 7: A curve and its inverse

depends on the parametrization of **C**. Since this parametrization does not appear explicitly in Eq'n (7) and hence it might be tempting to assume that just like the scalar line integral of Eq'n (2) this integral is also independent of the parametrization. Such, however, is not the case. Notice that whereas the vector field **F** is indenpedent of the parametrization, the unit tangent **t** has two possible values that depend on the sense in which the parametrization traverses **C**. Let us examine now the effect of reversing the traversal of the parametrization.

Let s be an arclength parameter of **C**, $a \leq s \leq b$. Then $\bar{s} = -s$ is also an arclength parameter of **C**. If **t** and $\bar{\mathbf{t}}$ denote the respective unit tangents, then

$$\bar{\mathbf{t}} = -\mathbf{t}$$

$$\mathbf{F} \cdot \bar{\mathbf{t}} = -\mathbf{F} \cdot \mathbf{t}$$

and hence the scalar integrals of $\mathbf{F} \cdot \bar{\mathbf{t}}$ and $\mathbf{F} \cdot \mathbf{t}$ are negatives of each other. Thus we have the following proposition:

Proposition 1.17 *The line integrals of two parametrizations of a curve are either equal or negatives of each other depending on whether they traverse the underlying curve in the same or opposite senses.*

□

An *oriented* curve is one for which a sense of traversal has been specified. This sense of traversal is called an *orientation* of the curve. Thus, every curve has two orientations. These orientations are designated by an arrow on the curve (Fig. 7). If **C** is an oriented curve, then the oppositely oriented curve is called the *inverse* of **C** and is denoted by $\mathbf{C}^{-1}$. It is clear that

$$\left(\mathbf{C}^{-1}\right)^{-1} = \mathbf{C}.$$

It follows from Proposition 1.17 that if **F** is a field whose domain contains the curve **C**then

$$\int_{\mathbf{C}^{-1}} \mathbf{F} \cdot \mathbf{ds} = -\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds} \qquad (8)$$

If **C** and **D** are two oriented curves such that the terminal point of **C** is also the initial point of **D** then their union is denoted by

$$\mathbf{C} + \mathbf{D}.$$

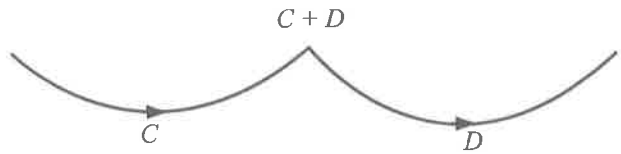


Figure 8: A curve and its inverse

It is clear that

$$\int_{\mathbf{C}+\mathbf{D}} \mathbf{F}\cdot\mathbf{ds} = \int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds} + \int_{\mathbf{D}} \mathbf{F}\cdot\mathbf{ds}. \tag{9}$$

EXERCISES 4.1

1. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y) = 1+2x-3y$ and $\mathbf{C}(t) = (1+2t, 2-3t), 0 \le t \le 3$.
2. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y) = 1+2x-3y$ and $\mathbf{C}(t) = (1+t^2, 2-t^3), 0 \le t \le 3$.
3. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y) = 1+x^2-y^3$ and $\mathbf{C}(t) = (t^2, t^3), 0 \le t \le 3$.
4. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y, z) = 1 + 2x - 3y + z$ and $\mathbf{C}(t) = (1+2t, 1+3t, 1+4t), 0 \le t \le 3$.
5. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y, z) = 1 + 2x - 3y + z$ and $\mathbf{C}(t) = (t^2, t^3, t), 0 \le t \le 3$.
6. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y, z) = 1 + 2x - 3y + z$ and $\mathbf{C}(t) = (1+2t, 1+3t, 1+4t), 0 \le t \le 3$.
7. Evaluate $\int_{\mathbf{C}} f ds$ where $f(x, y, z) = 1 + 2x - 3y + z$ and $\mathbf{C}$ is the line segment from (1, 2, 3) to (3, -2, -1).
8. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y) = (2x - 3y, x^2 + y)$ and $\mathbf{C}(t) = (1+2t, 1+3t), 0 \le t \le 1$.
9. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y) = (2x - 3y^2, x^2 - y)$ and $\mathbf{C}(t) = (t^2, t^3), 0 \le t \le 2$.
10. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y) = (\cos x + \cos y, \sin x \sin y)$ and $\mathbf{C}(t) = (1+2t, 1+3t), 0 \le t \le \pi$.
11. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y, z) = (2x-3y^2+z, x^2-y-2z, x+y-z)$ and $\mathbf{C}(t) = (t^2, t, t^3), 0 \le t \le 1$.
12. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y, z) = (2x-3y^2+z, x^2-y-2z, x+y-z)$ and $\mathbf{C}$ is the line segment from (1, 2, 3) to (3, -2, -1).
13. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y, z) = (2x-3y^2+z, x^2-y-2z, x+y-z)$ and $\mathbf{C}$ is the line segment from (3, -2, -1) to (1, 2, 3).
14. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y) = (2x - 3y, x^2 + y)$ and $\mathbf{C}$ is the polygonal line from (1, 1) to (1, 5) and from there to (6, 5).
15. Evaluate $\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds}$ where $\mathbf{F}(x, y) = (2x^2 - 3y, x + y^2)$ and $\mathbf{C}$ is the polygonal line from (1, 1) to (5, 1) and from there to (5,6).

16. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F}(x,y) = (2x - 3y, x + y)$ and $\mathbf{C}$ is the circle of radius 5 centered at the origin.

17. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F}(x,y) = (2x - 3y, x + y)$ and $\mathbf{C}$ is the circle of radius 5 centered at (1, -1).

18. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F}(x,y) = (2x - 3y, x + y)$ and $\mathbf{C}$ is counterclockwise ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

19. Evaluate $\int_{\mathbf{C}} (x^2+y)dx + (y-2x^2)dy$ where $\mathbf{C}(t) = (1+2t, 1+3t), 0 \leq t \leq 1$.

20. Evaluate $\int_{\mathbf{C}} (x^2 + y)dx + (y - 2x^2)dy$ where $\mathbf{C}(t)$ is the polygonal path from (2, 0) to (-2, 0) and from there to (-2, -4).

21. Evaluate $\int_{\mathbf{C}} \cos y dx + \sin 2x dy$ where $\mathbf{C}$ is unit circle of radius 1 centered at the origin.

22. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F}(x,y) = (2x - 3y, x^2 + y)$ and $\mathbf{C}(t) = (1 + 2t, 1 + 3t), 0 \leq t \leq 1$.

23. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F}(x,y) = (2x - 3y^2, x^2 - y)$ and $\mathbf{C}(t) = (t^2, t^3), 0 \leq t \leq 2$.

24. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F}(x,y) = (\cos x + \cos y, \sin x \sin y)$ and $\mathbf{C}(t) = (1 + 2t, 1 + 3t), 0 \leq t \leq \pi$.

25. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F}(x,y) = (2x - 3y^2, x^2 - y)$ and $\mathbf{C}(t) = (t^2, t^3), 0 \leq t \leq 1$.

26. Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F}(x,y) = (2x - 3y^2, x^2 - y)$ and $\mathbf{C}$ is the line segment from (1, 2) to (3, -2).

27. Evaluate the total flow of the field $\mathbf{F}(x,y) = (2x - 3y, x^2 + y)$ across the polygonal line from (1, 1) to (1, 5) and from there to (6, 5).

28. Evaluate the total flow of the field $\mathbf{F}(x,y) = (2x^2 - 3y, x + y^2)$ and $\mathbf{C}$ is the polygonal line from (1, 1) to (5, 1) and from there to (5,6).

29. Evaluate the total flow of the field $\mathbf{F}(x,y) = (2x - 3y, x + y)$ across the circle of radius 5 centered at the origin.

30. Evaluate the total flow of the field $\mathbf{F}(x,y) = (2x - 3y, x + y)$ across the circle of radius 5 centered at (1, -1).

31. Evaluate the total flow of the field $\mathbf{F}(x,y) = (2x - 3y, x + y)$ across counterclockwise ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

2 Surface integrals

We begin with the integration of scalar functions on a surface. Let $f : D \to \Re^3$ and let $\mathbf{S} \subset D$ be a parametrized surface in $\Re^3$. The points $\mathbf{S}(u,v), \mathbf{S}(u+\Delta u, v), \mathbf{S}(u+\Delta u, v+\Delta v), \mathbf{S}(u, v+\Delta v)$ form a near-parallelogram whose area is approximately

$$|(\mathbf{S}(u + \Delta u, v) - \mathbf{S}(u,v)) \times (\mathbf{S}(u, v + \Delta v) - \mathbf{S}(u,v))|$$

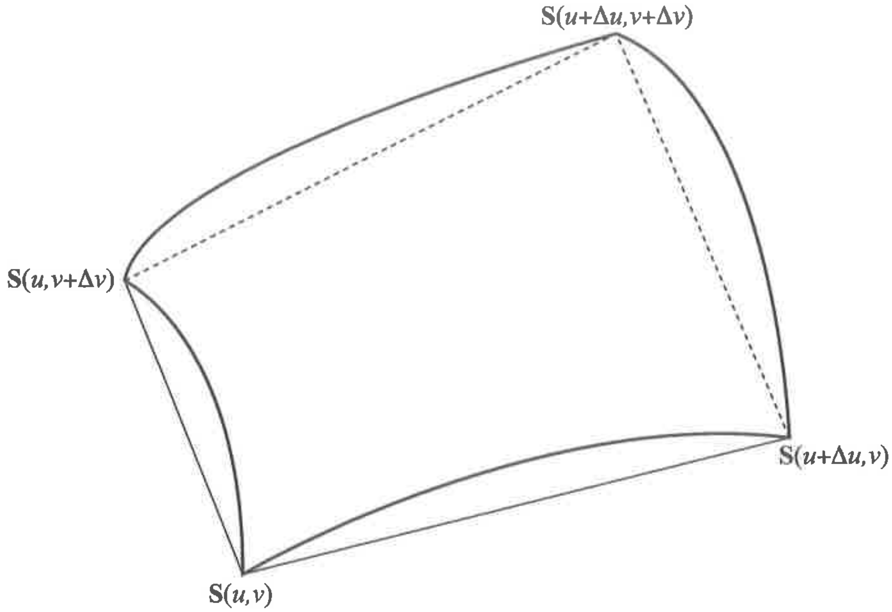


Figure 9: A surface element

$$\approx |\mathbf{S}_u \Delta u \times \mathbf{S}_v \Delta v| = |\mathbf{S}_u \times \mathbf{S}_v| \Delta u \Delta v,$$

which, by Lagrange's identity, equals

$$\sqrt{(\mathbf{S}_u \cdot \mathbf{S}_u)(\mathbf{S}_v, \mathbf{S}_v) - (\mathbf{S}_u \cdot \mathbf{S}_v)(\mathbf{S}_v \cdot \mathbf{S}_u)} \Delta u \Delta v = \sqrt{EG - F^2} \Delta u \Delta v$$

where

$$E = \mathbf{S}_u \cdot \mathbf{S}_u, \quad F = \mathbf{S}_u \cdot \mathbf{S}_v = \mathbf{S}_v \cdot \mathbf{S}_u, G = \mathbf{S}_v \cdot \mathbf{S}_v$$

Consequently

$$\int \int_{\mathbf{S}} f \mathbf{dS} = \int \int_D f \sqrt{EG - F^2} du dv. \tag{10}$$

Example 2.1 *The hemisphere of radius 2 that is centered at the origin and lies in the upper halfspace $z \geq 0$ has mass density $4 + x + y$ at the point above (x, y). Find the total mass of the hemisphere.*

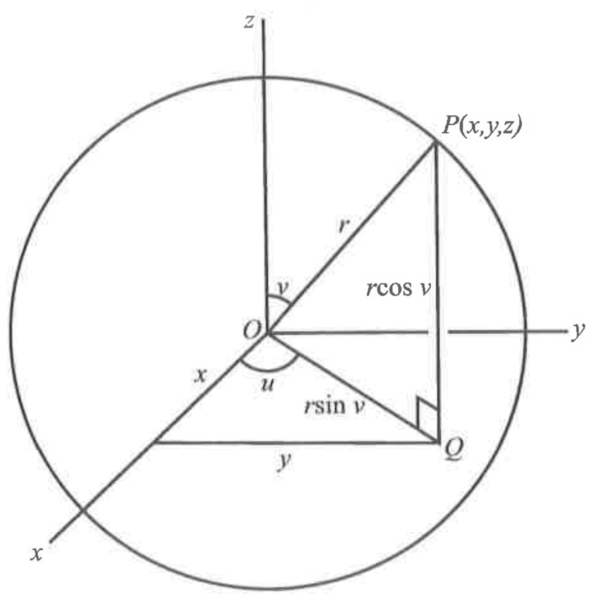
This hemisphere can be parametrized as (see Fig. 10)

$$\mathbf{S}(u, v) = 2(\cos u \sin v, \sin u \sin v, \cos v)$$

and so

$$\mathbf{S}_u = 2(-\sin u \sin v, \cos u \sin v, 0)$$

$$\mathbf{S}_v = 2(\cos u \cos v, \sin u \cos v, -\sin v).$$



$x = r\cos u \sin v$
$y = r\sin u \sin v$
$z = r\cos v$

Figure 10: A parametrization of a sphere of radius r

Thus,

$$E = \mathbf{S}_u \cdot \mathbf{S}_u = 4\sin^2 v$$
$$F = \mathbf{S}_u \cdot \mathbf{S}_v = 0$$
$$G = \mathbf{S}_v \cdot \mathbf{S}_v = 4$$
$$\sqrt{EG - F^2} = 4\sin v$$

Consequently the mass of the hemisphere is

$$\int_0^{\pi/2}\int_0^{2\pi}(4 + 2\cos u \sin v + 2\sin u \sin v)4\sin v\,du\,dv = 32\pi.$$

The surface which constitutes the graph of the function

$$z = f(x, y)$$

can be parametrized as

$$\mathbf{S}(x, y) = (x, y, f(x, y)).$$

Here

$$\mathbf{S}_x = (1, 0, f_x), \quad \mathbf{S}_y = (0, 1, f_y).$$

Consequently

$$E = \mathbf{S}_x \cdot \mathbf{S}_x = 1 + f_x^2$$
$$F = \mathbf{S}_x \cdot \mathbf{S}_y = f_x f_y$$
$$G = \mathbf{S}_y \cdot \mathbf{S}_y = 1 + f_y^2$$
$$\sqrt{EG - F^2} = \sqrt{(1 + f_x^2)(1 + f_y^2) - (f_x f_y)^2} = \sqrt{1 + f_x^2 + f_y^2}.$$

Example 2.2 *The surface* $z = x^2 + y^2, 0 \leq x, y \leq 2$ *has density function* $\delta(x, y) = 1 + x + y$. *Compute its total mass.*

Here

$$\sqrt{EG - F^2} = \sqrt{1 + (2x)^2 + (2y)^2}$$

The mass is

$$\int_0^2\int_0^2 (1 + x + y)\sqrt{1 + 4x^2 + 4y^2}\,dx\,dy = 42.35\ldots\ .$$

When $\mathbf{F}$ is a 3-dimensional vector field, it is no longer meaningful to ask for its flow across a curve. However, it is reasonable to wish to quantify the flow across a surface $\mathbf{S}$. By analogy with the discussion above, we need simply project $\mathbf{F}$ onto the unit normal of $\mathbf{S}$ and then integrate the length of this projection over the surface $\mathbf{S}$. Granted, this is a bit oversimplified. There is the issue of establishing this normal.

Referring to Figure 7, note that the normal $\mathbf{n}$ to the surface at $\mathbf{S}(u,v)$ is the limiting position of the normal to the triangle formed by $\mathbf{S}(u,v)$, $\mathbf{S}(u+\Delta u, v)$, and $\mathbf{S}(u, v+\Delta v)$. The normal to this triangle has the same direction as

$$[\mathbf{S}(u+\Delta u, v) - \mathbf{S}(u,v)] \times [\mathbf{S}(u, v+\Delta v) - \mathbf{S}(u,v)]$$

and, by the Mean Value Theorem its limiting position is

$$\mathbf{S}_u \times \mathbf{S}_v. \tag{11}$$

This means that the vector above is normal to (the tangent plane of $\mathbf{S}$) and we select it as the positive direction of the flow. Of course, care must still be exercised. While the line of direction of this vector is independent of the parametrization, its actual direction can vary by a factor of -1. In other words, the choice of another parametrization can result in reversing the normal.

Example 2.3

Consider the surface which consists of the graph of $z = x^2 + y^2$. As noted above it can be parametrized as $\mathbf{S}(x,y) = (x, y, x^2+y^2)$ so that

$$\mathbf{S}_x \times \mathbf{S}_y = (1, 0, 2x) \times (0, 1, 2y) = (-2x, -2y, 1).$$

At the origin this gives a normal of 1. However, the same surface can be parametrized as $\mathbf{S}(x,y) = (-x, y, x^2+y^2)$ in which case

$$\mathbf{S}_x \times \mathbf{S}_y = (-1, 0, 2x) \times (0, 1, 2y) = (-2x, 2y, -1).$$

This time the normal at the origin is (0, 0, -1).

Let $\mathbf{F} = (F_1, F_2, F_3)$ be a 3-dimensional field and let the surface $\mathbf{S}$ be parametrized as

$$\mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in D.$$

We define the *total flow (flux)* of $\mathbf{F}$ across $\mathbf{S}$ as

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{n} d\mathbf{S} = \int\int_D \mathbf{F} \cdot \frac{\mathbf{S}_u \times \mathbf{S}_v}{|\mathbf{S}_u \times \mathbf{S}_v|} \sqrt{EG - F^2} dudv$$

$$= \int\int_D \mathbf{F} \cdot (\mathbf{S}_u \times \mathbf{S}_v) dudv.$$

Example 2.4 *Compute the total flow of the field $\mathbf{F} = (x, x+y, x+y-z)$ across the upper hemisphere of radius 2 centered at the origin.*

Using the parametrization of Example 2.1, we get

$$\mathbf{S}_u \times \mathbf{S}_v = -4\sin v(\cos u \sin v, \sin u \sin v, \cos u)$$

$$\mathbf{F} \cdot (\mathbf{S}_u \times \mathbf{S}_v) = -4\sin v(\cos u \sin v, \sin u \sin v, \cos u)$$

$$\cdot(\cos u \cos v, \cos u \cos v + \sin u \cos v, \cos u \cos v + \sin u \cos v - \sin v)$$

$$\int_0^{\pi/2}\int_0^{2\pi} \mathbf{F} \cdot (\mathbf{S}_u \times \mathbf{S}_\mathbf{v}) du dv = -14\pi/3.$$

Note that the normal at $\mathbf{S}(0, \pi/2) = (2, 0, 0)$ is

$$(\mathbf{S}_u \times \mathbf{S}_v)(0, \pi/2) = (-4, 0, 0)$$

which is directed into the interior of the surface. Hence the total flow is $14\pi/3$ out.

There is a useful higher dimensional analog of Eq'n (5). If

$$\mathbf{S}(u, v) = ((x(u, v), y(u, v), z(u, v))$$

Then

$$\mathbf{S}_u \times \mathbf{S}_v = (y_u z_v - y_v z_u, z_u x_v - z_v x_u, x_u y_v - x_v y_u)$$

$$\mathbf{F} \cdot (\mathbf{S}_u \times \mathbf{S}_v) = (y_u z_v - y_v z_u)F_1 + (z_u x_v - z_v x_u)F_2 + (x_u y_v - x_v y_u)F_3$$

and so

$$\int\int_\mathbf{S} \mathbf{F} \cdot d\mathbf{S}$$

$$= \int\int_\mathbf{S} (y_u z_v - y_v z_u)F_1 du dv + \int\int_\mathbf{S} (z_u x_v - z_v x_u)F_2 du dv$$

$$+ \int\int_\mathbf{S} (x_u y_v - x_v y_u)F_3 du dv$$

We now evaluate the first of these three integrals by setting $u = y, v = z$ and obtain

$$\int\int_\mathbf{S} ((1 \cdot 1 - 0 \cdot 0)F_1 dy dz = \int\int_\mathbf{S} F_1 dy dz.$$

The other two integrals are evaluated by setting $u = z, v = x$ and $u = x, v = y$ respectively. These three evaluations yield

$$\int\int_\mathbf{S} \mathbf{F} \cdot d\mathbf{S} = \int\int_\mathbf{S} F_1 dy dz + F_2 dz dx + F_3 dx dy.$$

Example 2.5 *Compute the total flow of the field* $\mathbf{F}(x, y, z) = (x, x + y, x + y - z)$ *across the surface of the cube whose vertices all have coordinates 0 or 1 (Figure 11).*

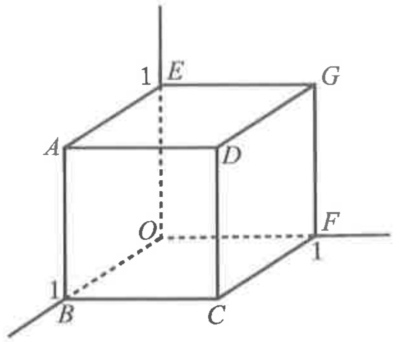


Figure 11: A cube

The faces $ABCD$ and $EOFG$ can be parametrized as $(1, y, z)$ and $(0, y, z)$ respectively, for both of which the normal is

$$\mathbf{S}_y \times \mathbf{S}_z = (1, 0, 0).$$

For these faces dx vanishes and hence the flows across these faces are

$$\int_0^1 \int_0^1 1 dydz = 1 \quad \text{and} \quad \int_0^1 \int_0^1 0 dydz = 0.$$

The faces $DCFG$ and $ABOE$ can be parametrized as $(x, 1, z)$ and $(x, 0, z)$ respectively, for both of which the normal is

$$\mathbf{S}_x \times \mathbf{S}_z = (0, -1, 0).$$

For these surfaces dy vanishes and hence the flows across these faces are

$$\int_0^1 \int_0^1 (x + 1) dxdz = \frac{3}{2} \quad \text{and} \quad \int_0^1 \int_0^1 (x + 0) dxdz = \frac{1}{2}.$$

The faces $ADGE$ and $BCFO$ can be parametrized as $(x, y, 1)$ and $(x, y, 0)$ respectively, for both of which the normal is

$$\mathbf{S}_x \times \mathbf{S}_y = (0, 0, 1).$$

For these surfaces dz vanishes and hence the flows across these surfaces are

$$\int_0^1 \int_0^1 (x + y - 1) dxdy = 0 \quad \text{and} \quad \int_0^1 \int_0^1 (x + y - 0) dxdy = 1.$$

Taking account of which normals point into the box and which point out, the total flow out of the box is

$$1 - 0 + \frac{3}{2} - \frac{1}{2} + 0 - 1 = 1.$$

EXERCISES 4.2

1. The hemisphere of radius 2 that is centered at the origin and lies in the upper halfspace $z \geq 0$ has mass density $x^2 + y^2$ at the point above (x, y). Find the total mass of the hemisphere.

2. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous Exercise.

3. The hemisphere of radius 1 that is centered at the origin and lies in the upper halfspace $z \geq 0$ has mass density $2 + xy$ at the point above (x, y). Find the total mass of the hemisphere.

4. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous Exercise.

5. Show that the surface area of the sphere of radius R is $4\pi R^2$.

6. Find a formula for the area of the lateral surface of a circular cylinder of height h and whose base has radius R.

7. Find a formula for the area of the lateral surface of a right circular cone of height h and whose base has radius R.

8. Find the area of the portion of the plane $x + 2y + 3z = 12$ in the first octant.

9. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous Exercise.

10. If the mass density is $\delta(x, y, z) = 1 + x + y + z$, find the total mass of the sphere of radius 1 centered at (1, 1, 1).

11. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous Exercise.

12. If the mass density is $\delta(x, y, z) = 1 + x + y + z$, find the total mass of the surface the cube whose vertices have coordinates 1 or 2.

13. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous exercise.

14. Find the area of the portion of the graph of $z = 4 - x^2 - y^2$ above the xy-plane.

15. Let $\mathbf{F} = (y - 2x, z - 2y, x - 2z)$. Compute the total flow of $\mathbf{F}$ across the surface of the previous Exercise.

3 Some Standard Parametrizations

Straight line:

$$\mathbf{x}(t) = \mathbf{a} + \mathbf{b}t = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t)$$

$$\mathbf{x}(t) = (1 - t)\mathbf{a} + t\mathbf{b} = ((1 - t)a_1 + tb_1, (1 - t)a_2 + b_2, (1 - t)a_3 + tb_3)$$

Planes parallel to coordinate planes:

$$\mathbf{S}(y, z) = (0, y, z), \qquad \mathbf{S}(x, z) = (x, 0, z), \qquad , \mathbf{S}(x, y) = (x, y, 0).$$

Sphere of radius R:

$$\mathbf{S}(u,v) = (R\cos u \sin v, R\sin u \sin v, R\cos v)$$

$$\mathbf{S}_u \times \mathbf{S}_v = -R^2(\cos u \sin v, \sin u \sin v, \cos v)$$

Surface of $z = f(x,y)$:

$$\mathbf{S}(x,y) = (x, y, f(x,y))$$
$$\mathbf{S}_x \times \mathbf{S}_y = (-f_x, -f_y, 1)$$

4 Summary

$$\int_{\mathbf{C}} f ds = \int_a^b f(\mathbf{C}(t))|\mathbf{C}'(t)|dt = \text{scalar line integral}$$

$$\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{ds} = \int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{t}ds = \int_{\mathbf{C}} F_1dx + F_2dy = \int_a^b \mathbf{F}\cdot\mathbf{C}'(t)dt$$
$$= \text{line integral, flow along curve, circulation, 2 or 3 variables}$$

$$\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{dn} = \int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{n}ds = \int_{\mathbf{C}} -F_2dx + F_1dy = \text{flow (flux) across } \mathbf{C}$$

$$\int\int_{\mathbf{S}} f\mathbf{dS} = \int\int_D f\sqrt{EG-F^2}dudv = \text{scalar surface integral}$$

$$\int\int_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS = \int\int_{\mathbf{S}} \mathbf{F}\cdot\mathbf{dS} = \int\int_D \mathbf{F}\cdot(\mathbf{S}_u \times \mathbf{S}_v)dudv$$
$$= \text{surface integral, flux across } \mathbf{S}$$

18 + 18 + 24