3. VECTOR DIFFERENTIATION
September 16, 2009

1 Limits

It is expected that the readers, who already have some calculus courses under their belts, have, at least, a working understanding of one dimensional limits, if not a theoretical one as well. In other words, given a "reasonable" function $f : D \subset \Re \rightarrow \Re$ they can evaluate the expression, or *limit*,

$$\lim_{x \to a} f(x).$$

They are probably aware that, informally speaking, the equation

$$\lim_{x \to a} f(x) = A$$

means that as a "becomes" A, $f(x)$ becomes "A". Thus,

$$\lim_{x \to -2} x^2 = 4$$

means that when x becomes -2, x^2 becomes 4. Less trivially,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

means that as x becomes 0, $(\sin x)/x$ becomes 1.

Moreover, the readers should be aware that this notion of limit has the following, by no means surprising, properties. If

$$\lim_{x \to a} f(x) = A \quad \text{and} \quad \lim_{x \to a} g(x) = B \tag{1}$$

then

$$\lim_{x \to a} [f(x) \pm g(x)] = A \pm B \tag{2}$$

$$\lim_{x \to a} [cf(x)] = cA \tag{3}$$

$$\lim_{x\to a}[f(x)g(x)] = AB \tag{4}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{A}{B} \quad \text{provided } B \neq 0. \tag{5}$$

The evaluation of two dimensional limits, however, gives rise to situations where intuition can be misleading. Consider, for example, the function

$$f(x,y) = \frac{2xy}{x^2+y^2}$$

One might be tempted to evaluate the limit of this function as $x \to 0$ and $y \to 0$ as

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right] = \lim_{y\to 0}\frac{2\cdot 0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0. \tag{6}$$

This might seem reasonable until one tries such points as

$$(x,y) = \left(\frac{1}{n},\frac{1}{n}\right)$$

because for such points, which clearly come arbitrarily close to the origin, we have

$$f\left(\frac{1}{n},\frac{1}{n}\right) = \frac{2\frac{1}{n}\frac{1}{n}}{\left(\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)^2} = 1,$$

which 1 is substantially different from the purported limit of 0 derived in (6). As the simplicity of this function f indicates, such problems are widespread and extra care must be exercised when defining the limit of a function of two or more variables.

Let $f : R \subset \Re^2 \to \Re$ be a function of two variables. Let $\mathbf{x} = (x,y)$ and $\mathbf{a} = (a,b)$. We say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = A$$

provided for each number $\epsilon > 0$, there exists a numbers $\delta > 0$ such that

$$|f(\mathbf{x}) - A| < \epsilon \quad \text{provided} \quad |\mathbf{x}-\mathbf{a}| < \delta.$$

This condition can be stated geometrically (Fig. 1) as

For any interval $I = (A-\epsilon, A+\epsilon)$ in the range of f there exists a disc D in the domain, that is centered at $\mathbf{a}$ and has radius δ such that

$$f(D) \subset I.$$

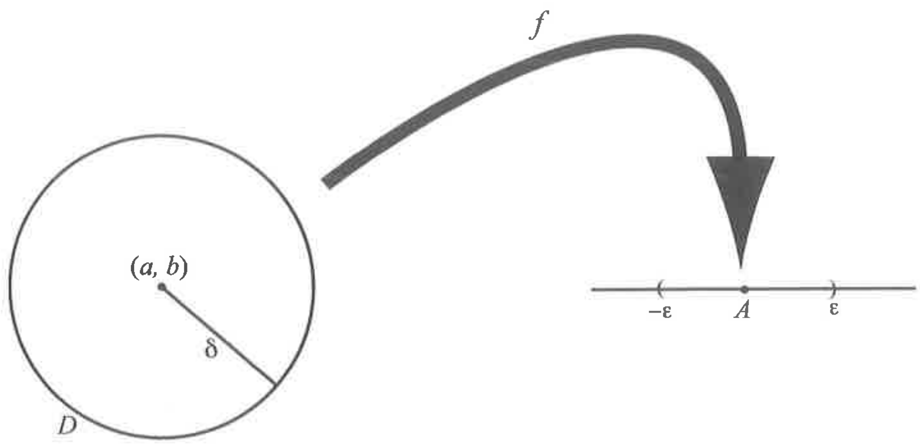

Figure 1: The geometry of the limit

Example 1.1 *If*

$$f(\mathbf{x}) = \frac{2x^2y^2}{x^2+y^2} \quad \textit{for} \quad \mathbf{x} \neq \mathbf{0}$$

show that

$$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$$

For every $\epsilon > 0$ we set

$$\delta = \sqrt{\epsilon}.$$

Clearly

$$|\mathbf{x} - \mathbf{a}| < \delta \quad \text{if and only if} \quad x^2 + y^2 < \delta^2$$

and then

$$|f(\mathbf{x}) - 0| = \left|\frac{2x^2y^2}{x^2+y^2} - 0\right| = \frac{2x^2y^2}{x^4+y^4}\frac{x^4+y^4}{x^2+y^2}$$

$$\leq 1 \cdot \frac{(x^2+y^2)^2}{x^2+y^2} \leq x^2+y^2 < \delta^2 = \epsilon.$$

Example 1.2 *If*

$$f(\mathbf{x}) = \frac{2xy}{x^2+y^2} \quad \textit{for} \quad \mathbf{x} \neq \mathbf{0}$$

show that

$$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) \neq 0$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \neq 0 \tag{7}$$

For $\epsilon = 0.9$ and for any $\delta > 0$ whatsoever, we can find a point $\mathbf{x}$ such that

$$|\mathbf{x} - \mathbf{0}| < \delta \quad \text{and} \quad \left|\frac{2xy}{x^2+y^2} - 0\right| \geq 0.9. \tag{8}$$

Given a function $f : D \subset \Re^3 \to \Re$ we define the *partial derivative* of f with respect to the component x_i of $\mathbf{x}$, at $\mathbf{a}$, to be

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h} \tag{10}$$

whenever this limit exists. Note that as $h \to 0$ the values of

$$\mathbf{a} + h\mathbf{e}_i$$

vary only in their i'th coordinate. Hence, the partial derivative of f is a derivative of the function obtained by holding all the variables except the i'th one constant, and differentiating with respect to x_i.

Example 3.1 *If*

$$f(x_1, x_2, x_3) = x_1 + x_2^2 x_3^4 + x_1 \sin(x_2 + 4x_3)$$

Evaluate

$$\frac{\partial f}{\partial x_i}(-1, 0, 1), \quad i = 1, 2, 3.$$

By definition

$$\frac{\partial f}{\partial x_1}(x_1, x_2, x_3) = 1 + 0 + 1 \cdot \cos(x_2 + 4x_3) = 1 + \sin(x_2 + 4x_3)$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2, x_3) = 2x_2 x_3^4 + x_1 \cos(x_2 + 4x_3)$$

$$\frac{\partial f}{\partial x_3}(x_1, x_2, x_3) = 4x_2^2 x_3^3 + 4x_1 \cos(x_2 + 4x_3)$$

and hence

$$\frac{\partial f}{\partial x_1}(-1, 0, 1) = 1 + \sin 4$$
$$\frac{\partial f}{\partial x_2}(-1, 0, 1) = -\cos 4$$
$$\frac{\partial f}{\partial x_3}(-1, 0, 1) = -4\cos 4.$$

Partial derivatives are derivatives and hence the usual differentiation rules hold for them. Namely,

Proposition 3.2 *Let $f, g : D \subset \Re^n \to \Re$ have partial derivatives at* $\mathbf{a}$. *Then*

1. $\frac{\partial(f \pm g)}{\partial x_i}(\mathbf{a}) = \frac{\partial f}{\partial x_i}(\mathbf{a}) \pm \frac{\partial g}{\partial x_i}(\mathbf{a})$

2. $\frac{\partial(fg)}{\partial x_i}(\mathbf{a}) = f(\mathbf{a})\frac{\partial g}{\partial x_i}(\mathbf{a}) + \frac{\partial f}{\partial x_i}(\mathbf{a})g(\mathbf{a})$

3. $\frac{\partial(\frac{f}{g})}{\partial x_i}(\mathbf{a}) = \frac{f(\mathbf{a})\frac{\partial g}{\partial x_i}(\mathbf{a}) - \frac{\partial f}{\partial x_i}(\mathbf{a})g(\mathbf{a})}{[g(\mathbf{a})]^2}$ provided $g(\mathbf{a}) \neq 0$.

□

It is also customary to use the following notations for the partial derivatives:

$$\frac{\partial f}{\partial x_i} = f_i$$

and, if $f = f(x, y, z)$ then

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x} = f_x, \quad \text{etc.}$$

The function $f : D \subset \Re^n \to \Re$ is said to be *differentiable* at $\mathbf{a}$ if it has partial derivatives at $\mathbf{a}$ and these partial derivatives are continuous functions.

EXERCISES 3.3

1. Give an example that distinguishes between function with partial derivatives and differentiable functions.

2. Explain why the fuss about *continuous* differentiability.

3. Evaluate the partial derivatives of the following functions at (1, -2, 3)

a. $f(x, y, z) = xy^2z^3$
b. $f(x, y, z) = e^{3x+4y+z}$
c. $f(x, y, z) = \sin(x + 2y)\cos(y - z)$
d. $f(x, y, z) = \frac{xy^2z^3}{3x+4y+z}$

4 Derivatives and Approximation

The Mean Value Theorem is one of the fundamental theorem of theoretical calculus. One way of stating this proposition is to say that it formalizes the imprecise but very useful approximation

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}. \tag{11}$$

Proposition 4.1 (Mean Value Theorem) *Suppose the function $f : [a, b] \to \Re$ is continuous and is also differentiable in (a, b). Then there exists a number $\theta, 0 < \theta < 1$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(a + \theta(b - a)). \tag{12}$$

□

A proof and further discussion of this theorem can be found in the Abstract Calculus appendix.

Lemma 4.2 *Let the function $f : D \subset \Re \to \Re$ have a derivative at $x \in D$. Then there exists a function $\delta = \delta(x, h)$ such that*

i. $f(x+h) = f(x) + hf'(x) + h\delta$;

ii. $\lim_{h \to 0} \delta = 0$.

PROOF: Define

$$\delta = \frac{f(x+h) - f(x)}{h} - f'(x).$$

Property i is satisfied for straightforward algebraic reasons. Property ii is justified by the existence of the derivative. Q.E.D.

Example 4.3 *Let $f(x) = x^3$. Find the function δ whose existence is guaranteed by Lemma 4.2.*

Following the paradigm set in the proof of Lemma 4.2,

$$\delta = \frac{(x+h)^3 - x^3}{h} - 3x^2 = 3xh + h^2.$$

Note that it is obvious that

$$\lim_{h \to 0} = 0.$$

Example 4.4 *Let $f(x) = \sin x$. Find the function δ whose existence is guaranteed by Lemma 4.2.*

Following the paradigm set in the proof of Lemma 4.2,

$$\delta = \frac{\sin(x+h) - \sin x}{h} - \cos x.$$

This time, it is not quite so obvious that the required limit is 0, but this is true nonetheless. This follows from the fact that

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = (\sin x)' = \cos x.$$

Lemma 4.5 *Let the function $f : D \subset \Re \to \Re$. Let A be a number and $\epsilon = \epsilon x, h$ be a function such that*

i. $f(x+h) = f(x) + hA + h\epsilon$;

ii. $\lim_{h \to 0} \epsilon = 0$.

Then $f'(x) = A$ and $\epsilon = \delta$ where δ is the function guaranteed by Lemma 4.2.

PROOF: Clearly

$$\frac{f(x+h)-f(x)}{h} = \frac{hA+h\epsilon}{h} = A+\epsilon.$$

Taking limits as $h \to 0$ yields

$$f'(x) = A.$$

The equality of δ and ϵ follows from parts i above and in Lemma 4.2. Q.E.D.

The next theorem generalizes Lemma 2 to a domain with two dimensions.

Theorem 4.6 *Let $f : D \subset \Re^2 \to \Re$. Then f is differentiable at (x,y) if and only if there exist continuous functions $A = A(x,y), B = B(x,y), \epsilon_1 = \epsilon_1(x,y,h,k), \epsilon_2 = \epsilon_2(x,y,h,k)$ such that*

$$f(x+h,y+k) = f(x,y) + hA + kB + h\epsilon_1 + k\epsilon_2$$

where

$$\lim_{(h,k)\to(0,0)} \epsilon_1 = 0 = \lim_{(h,k)\to(0,0)} \epsilon_2.$$

PROOF: Suppose first that such $A, B, \epsilon_1, \epsilon_2$ do indeed exist. Then

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{f(x,y)+hA+h\epsilon_1-f(x,y)}{h}$$

$$= \lim_{h\to 0}[A+\epsilon_1] = A$$

and, similalrly,

$$\lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{f(x,y)+kB+k\epsilon_2-f(x,y)}{k}$$

$$= \lim_{k\to 0}[B+\epsilon_2] = B.$$

Hence the partial derivatives of f exist at (x,y) and are equal to the A and B respectively. Since A and B were stipulated to be continuous, so are the partial derivatives.

Conversely, assume that the partial derivatives of f exist and are continuous. Then, clearly

$$f(x+h,y+k) - f(x,y)$$
$$= f(x+h,y+k) - f(x+h,y) + f(x+h,y) - f(x,y).$$

By the Mean Value Theorem, there exist numbers $0 < \sigma, \tau < 1$ such that

$$f(x+h,y) - f(x,y) = h\frac{\partial f}{\partial x}(x+\sigma h, y)$$

$$f(x+h,y+k) - f(x+h,y) = k\frac{\partial f}{\partial y}(x+h, y+\tau k).$$

Set

$$\epsilon_1 = \frac{\partial f}{\partial x}(x+\sigma h, y) - \frac{\partial f}{\partial x}(x,y)$$
$$\epsilon_2 = \frac{\partial f}{\partial y}(x+h, y+\tau k) - \frac{\partial f}{\partial y}(x,y)$$

The assumed continuity of the partial derivatives of f implies that

$$\lim_{h\to 0} \epsilon_1 = 0 = \lim_{h\to 0} \epsilon_2$$

Moreover

$$f(x+h,y+k) - f(x,y)$$
$$= h\frac{\partial f}{\partial x}(x+\sigma h, y) + k\frac{\partial f}{\partial y}(x+h, y+\tau k)$$
$$= h\left[\frac{\partial f}{\partial x}(x,y) + \epsilon_1\right] + k\left[\frac{\partial f}{\partial y}(x,y) + \epsilon_2\right].$$

The proof is completed by setting

$$A = \frac{\partial f}{\partial x}(x,y) \quad B = \frac{\partial f}{\partial y}(x,y).$$

□

EXERCISES 3.4

1. For each of the following functions, find the function δ whose existence is guaranteed by Lemma 4.2. In each case prove directly that $\lim_{h\to 0} \delta = 0$.
   a. $f(x) = x^2$
   b. $f(x) = x^4$
   c. $f(x) = \cos x$
   d. $f(x) = x^n$ where n is a positive integer
   e. $f(x) = \sqrt{x}$.

5 The Chain Rule

Proposition 5.1 *Let $f(\mathbf{x}) : E \subset \Re^2 \to \Re$ and $\mathbf{x}(t) : D \subset \Re \to \Re^2$ be functions such that $f \circ \mathbf{x}$ is defined on D. If $\mathbf{x}(t)$ is differentiable at t and f is differentiable at $\mathbf{x}(t) = (x(t), y(t))$, then $f \circ \mathbf{x}$ is differentiable at t and*

$$\frac{d(f \circ \mathbf{x})}{dt} = \frac{\partial f}{\partial x}(x,y)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x,y)\frac{dy}{dt}.$$

PROOF: Let Δt be an arbitrary (small) number. Set

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Then, by two applications of the Mean Value Theorem,

$$\frac{\Delta f}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)}{\Delta t}$$

$$= \frac{\frac{\partial f}{\partial x}(x + \sigma\Delta x, y + \Delta y)\Delta x + \frac{\partial f}{\partial x}(x, y + \tau\Delta y)\Delta y}{\Delta t}$$

$$= \frac{\partial f}{\partial x}(x + \sigma\Delta x, y + \Delta y)\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial x}(x, y + \tau\Delta y)\frac{\Delta y}{\Delta t}.$$

Hence, because of the continuity of the partial derivatives, and the fact that

$$\lim_{\Delta t \to 0} \Delta x = 0 = \lim_{\Delta t \to 0} \Delta y,$$

it follows that

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}(x,y)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x,y)\frac{dy}{dt}.$$

Q.E.D.

Example 5.2 *Let $f(x, y)$ be a differentiable function such that*

$$f_x(5, 10) = 4, \quad f_y(5, 10) = -3.$$

Let $g(t) = f(1 - 2t, -2 + 3t^2)$. Evaluate $g'(-2)$.

Set $\mathbf{x}(t) = (1 - 2t, -2 + 3t^2)$ and note that

$$g(t) = (f \circ \mathbf{x})(t) \quad \text{and} \quad \mathbf{x}(-2) = (5, 10).$$

By Theorem 3.5.1

$$g'(t) = (f \circ \mathbf{x})'(t) = \frac{\partial f}{\partial x}(5,10)\frac{dx}{dt}(-2) + \frac{\partial f}{\partial y}(5,10)\frac{dy}{dt}(-2)$$

$$= 4 \cdot (-2) + (-3) \cdot (-12) = 28.$$

The proof of the following 3-dimensional analog of Proposition 5.1 is relegated to Exercise 5.

Proposition 5.3 *Let $f(\mathbf{x}) : E \subset \Re^3 \to \Re$ and $\mathbf{x}(t) : D \subset \Re \to \Re^3$ be functions such that $f \circ \mathbf{x}$ is defined on D. If $\mathbf{x}(t)$ is differentiable at t and f is differentiable at $\mathbf{x}(t) = (x(t), y(t), z(t))$, then $f \circ \mathbf{x}$ is differentiable at t and*

$$\frac{d(f \circ \mathbf{x})}{dt} = \frac{\partial f}{\partial x}(x,y,z)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x,y,z)\frac{dy}{dt} + \frac{\partial f}{\partial z}(x,y,z)\frac{dz}{dt}$$

EXERCISES 3.5

1. Let $f(x,y)$ be a differentiable function such that

$$f_x(1,0) = -4, \quad f_y(1,0) = 3.$$

Let $g(t) = f(3-2t^2, 2+2t)$. Evaluate $g'(-1)$.

2. Let $f(x,y)$ be a differentiable function such that

$$f_x(1,1) = 7, \quad f_y(1,1) = -1.$$

Let $g(t) = f(t^2, t^3)$. Evaluate $g'(1)$.

3. Let $f(x,y,z)$ be a differentiable function such that

$$f_x(5,10,6) = 4, \quad f_y(5,10,6) = -3, \quad f_z(5,10,6) = 2.$$

Let $g(t) = f(1-2t, -2+3t^2, t^2-t)$. Evaluate $g'(-2)$.

4. Let $f(x,y,z)$ be a differentiable function such that

$$f_x(1,1,1) = 2, \quad f_y(1,1,1) = -3, \quad f_z(1,1,1) = 2.$$

Let $g(t) = f(t, t^2, t^4)$. Evaluate $g'(1)$.

5. Prove Proposition 5.3.

6 The Gradient

Given a differentiable function $f : D \subset \Re^3 \to \Re$, we define the *nabla* symbol by its operation on f:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

This vector is called the *gradient of f.*

Example 6.1 *The gradient of $f(x,y,z) = xy^2z^3$ is $(y^2z^3, 2xyz^3, 3xy^2z^2)$.*

This concept has a variety of uses. Note first that it simplifies the chain rule of Proposition 3. 5 to the form

$$\frac{d(f \circ g)}{dt} = (\nabla f) \cdot \mathbf{x}.$$

Next comes the directional derivative. Consider the function

$$w = f(x,y,z) = 1 + 2x + 3y - 5z$$

at the origin $O = (0, 0,0)$. If the variable point V moves through O along the x-axis from, $x = -4$ to $x = 4$, then the corresponding value of f, namely $f(x, 0, 0) = 1 + 2x$, increases at the rate of 2 units of w per unit of x. Similarly, If the variable point V moves along the y-axis from, say, $y = -5$ to $y = 5$, then the corresponding value of w, namely $f(0, y, 0) = 1 + 3y$, increases at the rate of 3 units of z for each unit of y. If the variable point V moves along the z-axis from, say, $z = -6$ to $z = 6$, then the corresponding value of w, namely $f(0, 0, z) = 1 - 6z$, decreases at the rate of -5 units of w for each unit of z.

Finally, If V is constrained to travel along the line

$$x = y = z$$

then the corresponding value of w, namely

$$f(x, x, x) = 1 + 2x + 3x - 5x = 1$$

does not change at all. This illustrates the notion of the *directional derivative.* Given a function $f : D \subset \Re^3 \to \Re$ and a fixed point P in D and a unit vector $\mathbf{u}$ in $\Re^3$, we define

$$D_{\mathbf{u}}f = \lim_{h \to 0} \frac{f(\vec{OP} + h\mathbf{u}) - f(\vec{OP})}{h}.$$

to be the directional derivative of f along $\mathbf{u}$

Since $\vec{OP}$ and $\mathbf{u}$ are held constant then

$$f(\vec{OP} + t\mathbf{u})$$

can be regarded as the composite function

$$f(\mathbf{x}) \circ \mathbf{x}(t) \quad \text{where} \quad \mathbf{x}(t) = (p_1 + tu_1, p_2 + tu_2, p_3 + tu_3)$$

and then

$$D_{\mathbf{u}}f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} u_i = (\nabla f) \cdot \mathbf{u}.$$

Thus we have proved the following proposition:

Proposition 6.2 *If $f : D \subset \Re^3 \to \Re$ and $\mathbf{u}$ is a unit n-vector, then*

$$D_{\mathbf{u}}f = (\nabla f) \cdot \mathbf{u}$$

□

Example 6.3 *Let $f(x, y, z) = xy^2z^3 - x - 2y - 3z^2$ and $\mathbf{u} = (1, -1, 1)/\sqrt{3}$. Compute $D_{\mathbf{u}}f$ at (-1,0,1).*

Here

$$\nabla f = (y^2z^3 - 1, 2xyz^3 - 2, 3xy^2z^2 - 6z)$$
$$\nabla f(-1,0,1) = (-1,-2,-6)$$

and hence

$$D_{\mathbf{u}}f(-1,0,1) = (-1,-2,-6)\cdot\frac{(1,-1,1)}{\sqrt{3}} = \frac{-5}{\sqrt{3}}.$$

It is natural to ask for those directions (unit vectors $\mathbf{u}$) that maximize and minimize their directional derivatives. Let θ be the angle between ∇f and a unit vector $\mathbf{u}$ in $\Re^3$. By Proposition 6.2

$$D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta.$$

Since $-1 \leq \cos\theta \leq 1$ the next proposition follows immediately.

Proposition 6.4 *If $f : D \subset \Re^n \to \Re$ then*
i. $\mathbf{u}$ maximizes the directional derivative of f if and only if

$$\mathbf{u} = \frac{\nabla f}{|\nabla f|}$$

and this maximum value is $|\nabla f|$;
ii. $\mathbf{u}$ minimizes the directional derivative of f if and only if

$$\mathbf{u} = -\frac{\nabla f}{|\nabla f|}$$

and this minimum value is $-|\nabla f|$.

□

Example 6.5 *Let $f(x,y,z) = xy^2z^3 - x - 2y - 3z^2$. Find the largest and smallest values of the directional derivatives of f at (-1, 0, 1).*

It was seen in Example 6.3 that

$$\nabla f(-1,0,1) = (-1,-2,-6)$$

Hence the maximum and minimum directional derivatives are

$$\pm\sqrt{1^2+2^2+6^2} = \pm\sqrt{41}.$$

Let $z = f(x,y)$ be a function of two variables with graph S. As a variable point $V = (x,y)$ moves along a straight line m through a fixed point $P = (p_1,p_2)$ in the domain of f (i.e., the xy-plane), the corresponding point

$$Q = (x,y,f(x,y))$$

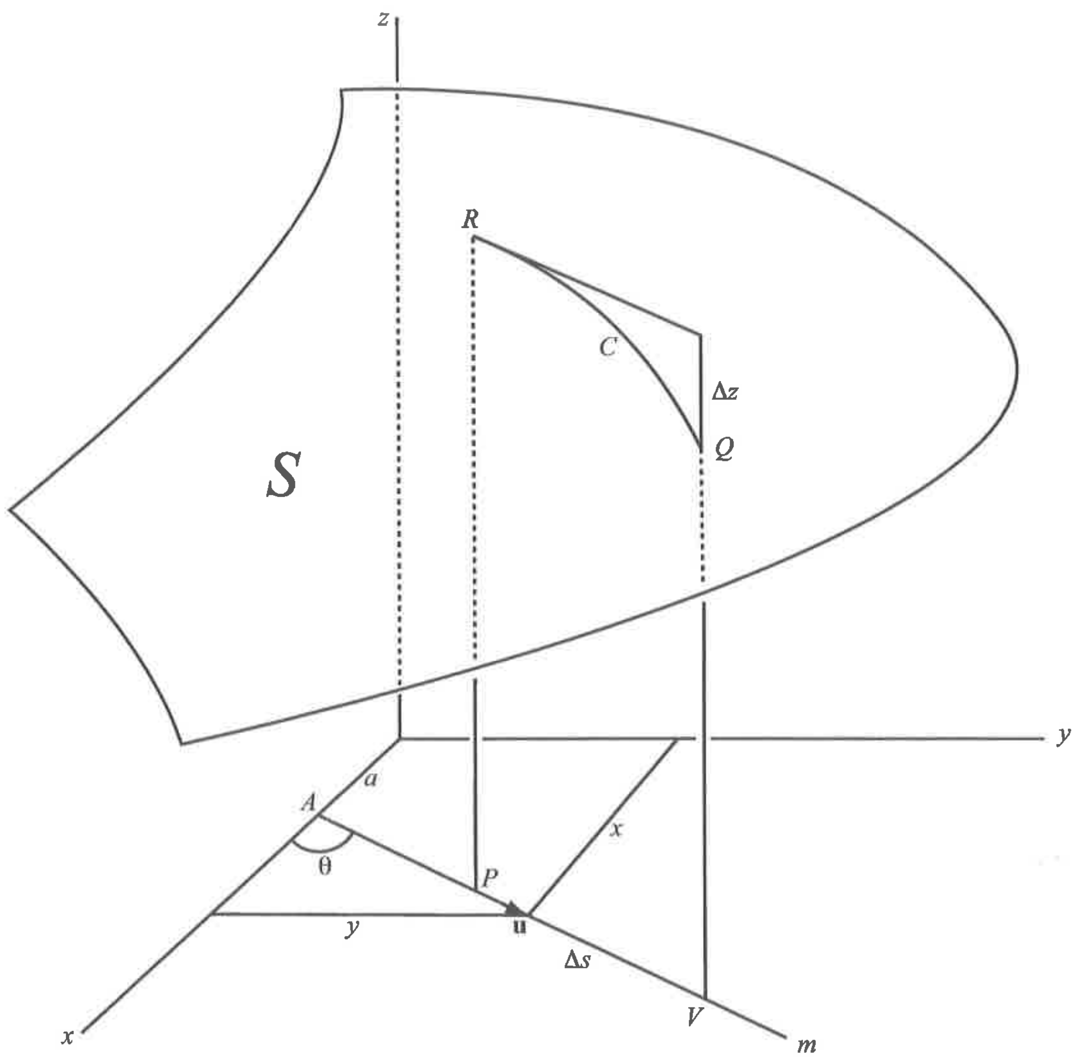

Figure 2: An interpretation of the directional derivative

is constrained to move on the surface S so as to stay above V (Figure 2). Thus Q moves along the cross section C of S with the vertical plane containing m. Let $\mathbf{u}$ be the unit vector from $A = (a, 0)$ in the direction of V, let θ be the angle, from the positive x-axis to m, and let s denote the (variable) distance from A to V. Note that the straight line m has the parametrization

$$x = a + s\cos\theta, \quad y = s\sin\theta.$$

Then the slope of C at R is

$$\lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s} = \frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = (\nabla f)\cdot\mathbf{u} = D_{\mathbf{u}}f.$$

Hence we have a new interpretation of the directional derivative.

Proposition 6.6 *Given a function $z = f(x,y)$, the slope of the cross section of its graph with a vertical plane is given by*

$$D_{\mathbf{u}}f$$

where $\mathbf{u}$ *is a unit vector parallel to the cutting plane.*

□

Example 6.7 *Water is flowing freely down the surface $z = x^2 + 2xy - y^3$. In what direction is it flowing above the point (1, -1) in the xy-plane? Also find the actual direction of flow in $\Re^3$.*

The water will flow in the direction that minimizes the directional derivative. The gradient is $(2x + 2y, 2x - 3y^2)$ evaluated at (1, -1), or (4, -1). Thus the direction of the water (projected to the xy plane) direction is

$$\mathbf{u} = \frac{(4, -1)}{\sqrt{17}}.$$

The actual flow in $\Re^3$ is along the vector

$$\left(\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}}, -\sqrt{17}\right) \quad \text{or} \quad (4, -1, -17)$$

anchored at (1, -1, 0).

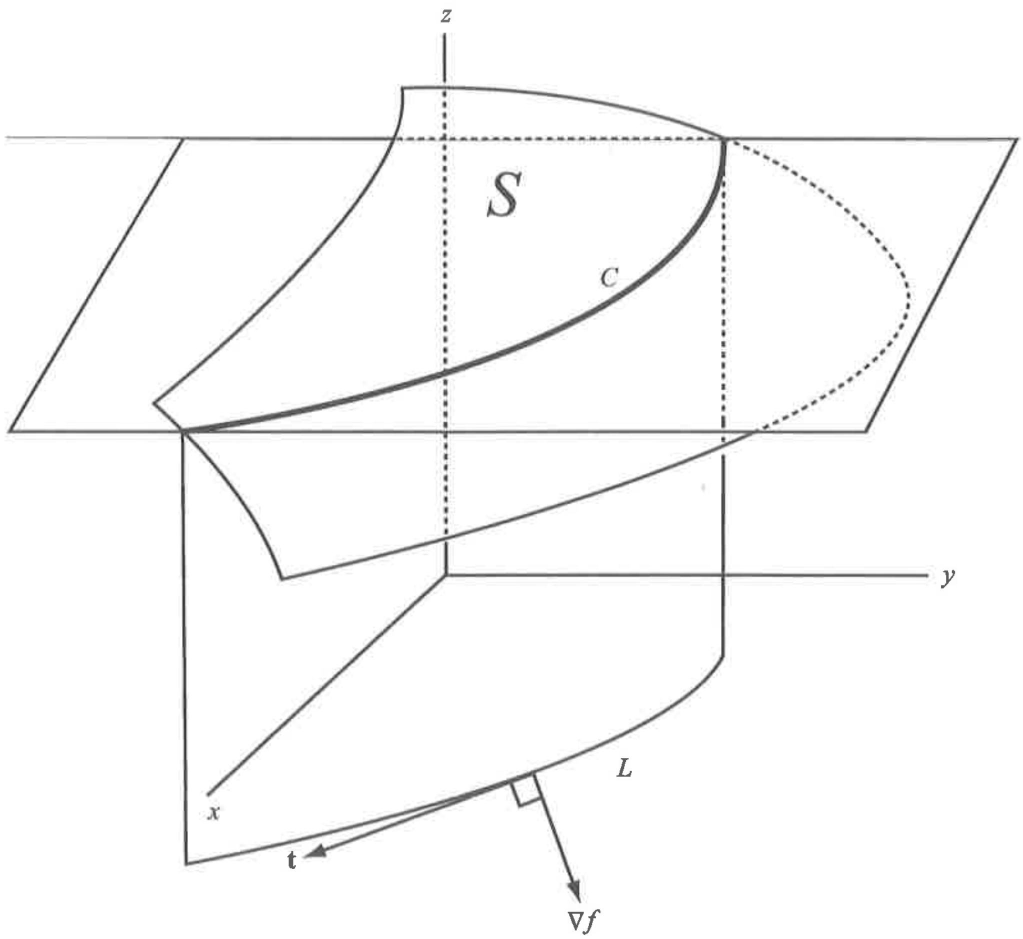

Figure 3: A level curve

Once again, let $z = f(x, y)$ be a function of two variables with graph S. Let C be a cross section of S with a horizontal plane $z = c$. Its projection L into the xy-plane (Figure 3) has equation

$$f(x, y) = c \tag{13}$$

and is called a *level curve.* Being a curve, C can also be parametrized as

$$x = x(t), \quad y = y(t).$$

When Eq'n (13) is differentiated with respect to t, the chain rule yields

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{dc}{dt} = 0$$

or

$$(\nabla f) \cdot \mathbf{t} = 0$$

where

$$\mathbf{t} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

is a vector tangent to the level curve L at the point (x, y).

Example 6.8 *Find the equations of the tangent and normal to the curve $x^2 + 2xy - y^3 = 0$ at the point $(1, -1)$.*

The normal has the same direction as the gradient $(4, -1)/\sqrt{17}$ (see Example 6.7). hence the normal has slope -1/4 and the tangent has slope 4. Thus the required equations are

$$y + 1 = -(x - 1)/4 \quad \text{and} \quad y + 1 = 4(x - 1).$$

Let S be the surface defined implicitly by the equation

$$f(x, y, z) = 0.$$

If C (Fig. 13) is any curve on the surface then it has a parametrization

$$(x(t), y(t), z(t))$$

such that

$$f(x(t), y(t), z(t)) = 0.$$

When this equation is differentiated with respect to t, the chain rule yields

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

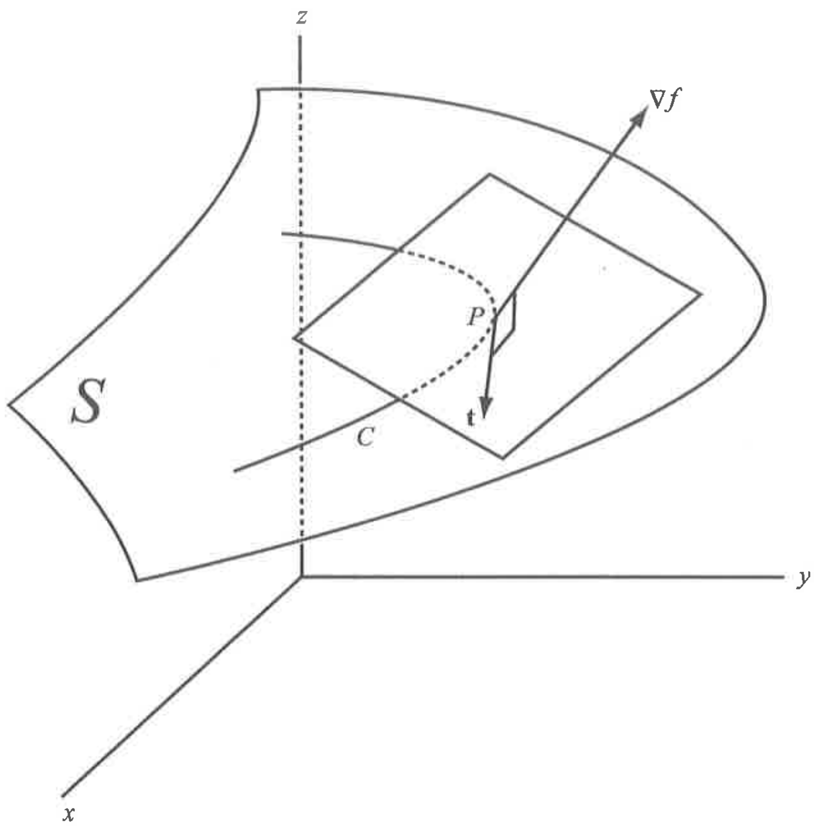

Figure 4: The formation of a tangent plane

or

$$(\nabla f) \cdot \mathbf{t} = 0$$

where

$$\mathbf{t} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

is a vector tangent to C at (x, y, z). It follows that all the tangent vectors to all the curves on the surface S that pass through the point P lie in the plane that is tangent to the point P. In other words,

Proposition 6.9 $(\nabla f)(P)$ *is the normal to the plane that is tangent to the surface*

$$f(x, y, z) = 0$$

at the point P.

Example 6.10 *Find the equation of the tangent plane to the surface* $x^2 + 2xyz^3 - 3yz = 0$ *at the point* (1,1,1).

The general gradient is $(2x+2yz^3, 2xz^3-3z, 6xyz^2-3y)$ which assumes the value

$$(4,-1,3))$$

at the point (1,1,1). Hence the required equation is

$$4(x-1)-(y-1)+3(z-1)=0.$$

EXERCISES 3.6

1. Find the gradient of $f(x,y)=x^3y^2+x^2y^5$ at (-1, 1).
2. Find the gradient of $f(x,y,z)=x^3y^2z^3+x^2y^5z^4$ at (-1, 1, 1).
3. Let $f(x,y,z)=x^3y^2z^3+x^2y^5z^4$ and $\mathbf{u}=(-1,1,1)/\sqrt{3}$. Compute $D_{\mathbf{u}}f$ at (-1,0,2).
4. Let $f(x,y,z)=xy^2z^3-x-2y-3z^2$. Find the largest and smallest values of the directional derivatives of f at (-1, 1,2).
5. Water is flowing freely down the surface $z=xy-x^3+xy^2$. In what direction is it flowing above the point (-1, 1) in the xy-plane? Also find the actual direction of flow in $\Re^3$.
6. Find the equation of the tangent plane to the surface $2x^2-3xyz^3+3y^2z=0$ at the point *(1,-1,1)*.

7 Functions

A vector function $\mathbf{F}(x): D \subset \Re \to \Re^3$ is said to be differentiable if each of its components is a differentiable function of x and we write

$$\mathbf{F}'(x)=\frac{d\mathbf{F}}{dx}=(F_1'(x), F_2'(x), F_3'(x)).$$

This operation satisfies the rules

$$(\mathbf{F}+\mathbf{G})'=\mathbf{F}'+\mathbf{G}'$$

$$(k\mathbf{F})'=k\mathbf{F}'$$

$$(f\mathbf{F})'=f'\mathbf{F}+f\mathbf{F}'$$

$$(\mathbf{F}\cdot\mathbf{G})'=\mathbf{F}'\cdot\mathbf{G}+\mathbf{F}\cdot\mathbf{G}'$$

$$(\mathbf{F}\times\mathbf{G})'=\mathbf{F}'\times\mathbf{G}+\mathbf{F}\times\mathbf{G}'$$

If $\mathbf{F}: D \subset \Re^n \to \Re^3$ is a function of two or more variables $x, y, \ldots$, then

$$\frac{\partial\mathbf{F}}{\partial x}=\mathbf{F}_x=\left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial x}, \frac{\partial F_3}{\partial x}\right)$$

$$\frac{\partial \mathbf{F}}{\partial y} = \mathbf{F}_y = \left(\frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial y}\right)$$

and so on.

Example 7.1

EXERCISES 3.7