# HYPERBOLIC CENTROIDS OF SOME REGIONS*

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January 17, 2006

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## ABSTRACT

Explicit expressions for the centroids of hyperbolic pie shapes and isosceles triangles are found and compared to their Euclidean analogs.

## 1

Interest in the concepts of moment and center of mass of two point-mass systems in non-Euclidean geometries goes back to to the 1870's [1, 2]. Centers of mass of finite point systems in the context of spherical geometry were defined in 1947 [3, 4]. The general issue of finite point-mass systems in hyperbolic, elliptic, and Euclidean spaces [5, 6] was resolved by Gal'perin only relatively recently [5, 6]. It was demonstrated there that the center of mass of a finite point-mass system has a very elegant description in the Minkowski model of hyperbolic geometry. An excellent exposition of this model can be found in [8], and a short, necessarily incomplete, summary of the relevant facts is given here.

For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\Re^3$ let

$$\mathbf{x} \circ \mathbf{y} = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

The underlying set $H^2$ of the model is the hyperboloid sheet

$$\{\mathbf{x} \in \Re^3 \mid \mathbf{x} \circ \mathbf{x} = -1, x_1 > 0\}$$

*This research was supported in part by University of Kansas General Research Allocation 2301559-003

If $\mathbf{x}, \mathbf{y} \in H^2$ then the hyperbolic distance between them is

$$d_H(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\mathbf{x} \circ \mathbf{y}) \tag{1}$$

The geodesics of this model are its intersections with the Euclidean planes that contain the origin of $\Re^3$. Let $p$ and $q$ be the geodesics determined by the planes normal to the vectors $\vec{u}$ and $\vec{v}$ respectively. Then the angle formed by $p$ and $q$ is defined by

$$\angle(p, q) = \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\sqrt{(\vec{u} \circ \vec{u})(\vec{v} \circ \vec{v})}}\right)$$

Let $A_1, A_2, ..., A_k$ be points of $H^2$ and let $m_1, m_2, ..., m_k$ be non-negative real numbers. Then $\chi = \{(A_i, m_i)\}_{i=1}^k$ is a *finite point-mass system.* The *center of mass* of $\chi$ is that point $C = C(\chi) \in H^2$ such that, if $O$ denotes the origin of $\Re^3$, then

$$m \cdot \vec{OC} = \sum_{i=1}^{k} m_i \, \vec{OA_i}$$

for some real number $m$. The number $m$ is interpreted as the total mass of the system $\chi$. In the two point case the center of mass $C$ is that point on the geodesic joining $A_1$ and $A_2$ such that

$$m_1 \sinh CA_1 = m_2 \sinh CA_2$$

and the total mass of the system is

$$m = m_1 \cosh CA_1 + m_2 \cosh CA_2$$

These equations were known in essence over a century ago [1, 2]. The center of mass of a uniform 3 point-mass system coincides with the point of intersection of the medians of the underlying hyperbolic triangle. This fact was observed in [10] in a rather specialized language and so we take this opportunity to offer an alternative proof. Let $E, F$ be the respective midpoints of the sides $AC, AB$ of $\Delta ABC$ (Fig. 1). Then the center of mass of $\{(A, w), (B, w)\}$ is the point-mass

$$(F, 2w \cosh c)$$

and hence the center of mass of $\{(A, w), (B, w), (C, w)\}$ lies on the point $M$ of $CF$ such that

$$2w \cosh c \sinh d_1 = w \sinh d_2$$

It follows that

$$\begin{aligned}\frac{AE}{EC}\frac{CM}{MF}\frac{FB}{BA} &= \frac{\sinh b}{\sinh b}\frac{\sinh d_2}{\sinh d_1}\frac{\sinh c}{\sinh 2c} \\ &= \frac{1}{1}\frac{2w \cosh c}{w}\frac{1}{2\cosh c} = 1\end{aligned}$$

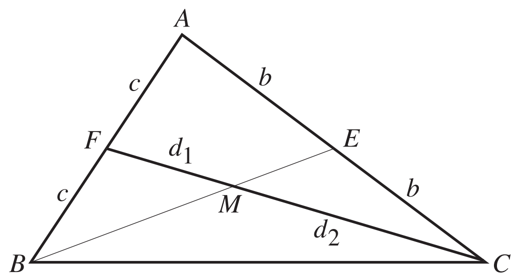


Figure 1:

Hence, by the converse to the theorem of Menelaus, the points $B, M$, and $E$ are collinear. Since the medians of the hyperbolic triangle are concurrent, their common intersection is also the center of mass in question.

The model $H^2$ is endowed with a polar coordinate-like parametrization as follows. Let $P = (1, 0, 0)$. For any point $\mathbf{x} \in H^2, \eta = \eta(\mathbf{x})$ is its hyperbolic distance from $P$ whereas $\theta = \theta(\mathbf{x})$ is the counterclockwise angle from the positive $x_2$ axis to the ray from the origin to the point $(0, x_2, x_3)$. Then (see [8, p. 88]),

$$\begin{cases} x_1 = \cosh \eta \\ x_2 = \sinh \eta \cos \theta \\ x_3 = \sinh \eta \sin \theta \end{cases} \tag{2}$$

Moreover, the area element with respect to this parametrization is

$$\sinh \eta d\eta d\theta$$

It is therefore natural to define the *centroid* of the region $R \subset H^2$ as that point $C$ of $H^2$ such that $\vec{OC}$ is codirectional with

$$\int \int_R (x_1, x_2, x_3) \sinh \eta d\eta d\theta \tag{3}$$

We refer to the vector in (3) as the *precentroid* of $R$.

Let $\Pi_{r,\alpha}$ denote the hyperbolic pie of radius $r$ and central angle $2\alpha$. Because of the transitivity of the hyperbolic plane and all of its models, it may be assumed that $\Pi_{r,\alpha}$ is positioned as in Figure 2. It is easily verified that the hyperbolic angle between the geodesics $PA$ and $PB$ is indeed $2\alpha$. Let $C$ denote the centroid of $\Pi_{r,\alpha}$. Equations (2) and (3) imply that $\vec{OC}$ is codirectional with the vector

$$\int_0^r \int_{-\alpha}^{\alpha} (\sinh \eta \cosh \eta, \sinh^2 \eta \cos \theta, \sinh^2 \eta \sin \theta) d\theta d\eta$$

$$= \left( \alpha \sinh^2 r, \frac{\sin \alpha}{2} (\sinh 2r - 2r), 0 \right) \tag{4}$$

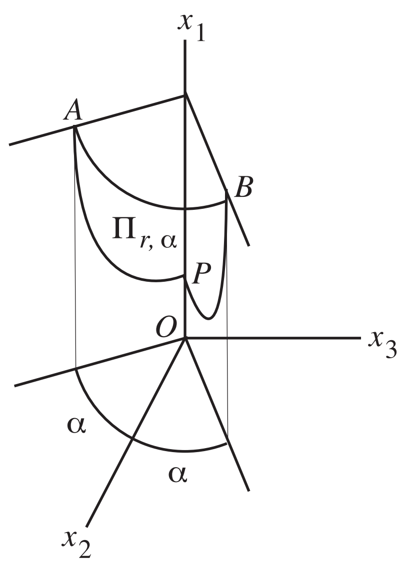


Figure 2:

**Proposition 1.1** *The distance $d(\Pi_{r,\alpha})$ from the vertex of the hyperbolic pie $\Pi_{r,\alpha}$ to its centroid is*

$$\tanh^{-1}\left(\frac{\sin\alpha(\sinh 2r - 2r)}{2\alpha\sinh^2 r}\right)$$

Proof: Let $(c_1, c_2, 0)$ denote the precentroid of $\Pi_{r,\alpha}$ given in (4). Then

$$d_H(C,P) = \cosh^{-1}\left(-\frac{(c_1,c_2,0)}{\sqrt{c_1^2 - c_2^2}} \circ (1,0,0)\right) = \cosh^{-1}\left(\frac{c_1}{\sqrt{c_1^2 - c_2^2}}\right)$$

It follows from the identity

$$\cosh^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \tanh^{-1} x \tag{5}$$

that

$$d(\Pi_{r,\alpha}) = \tanh^{-1}\left(\frac{\sin\alpha(\sinh 2r - 2r)}{2\alpha\sinh^2 r}\right)$$

Q.E.D.

Note that

$$\lim_{r\to\infty} d\left(\Pi_{r,\alpha}\right) = \tanh^{-1}\left(\frac{\sin\alpha}{\alpha}\right)$$

which is finite, in contrast with the Euclidean analog where the distance in question is

$$\frac{2r\sin\alpha}{3\alpha}$$

which is clearly unbounded. On the other hand, since

$$d(\Pi_{r,\alpha}) = \frac{2r\sin\alpha}{3\alpha} + O(r^3)$$

it follows that

$$\lim_{r\to 0}\frac{1}{r}d\left(\Pi_{r,\alpha}\right) = \frac{2\sin\alpha}{3\alpha}$$

which is consistent with the fact that infinitesimal hyperbolic regions are Euclidean.

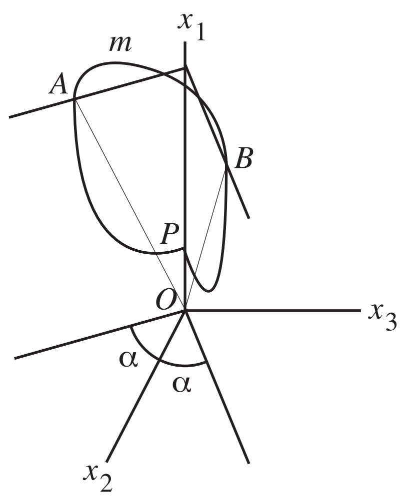


Figure 3:

Next we turn to isosceles triangles. In Figure 3, $A = (\cosh c, \cos\alpha\sinh c, \sin\alpha\sinh c)$, $B = (\cosh\eta, \cos\alpha\sinh c, -\sin\alpha\sinh c)$ and arc $AmB$ is the intersection of the hyperboloid $H^2$ with the plane $OAB$ . Then $\Delta PAB$ on $H^2$ has vertex angle $\angle APB = 2\alpha$ and equal sides $PA = PB$ of hyperbolic length $c$. Let $D$ be the midpoint of the side $AB$ and let $a$ and $b$ be the hyperbolic lengths of $AD = DB$ and $PD$ respectively. It follows from the trigonometry of the hyperbolic right triangle [7, 9] that

$$\sinh a = \sin\alpha\sinh c \qquad \text{and} \qquad \tanh b = \cos\alpha\tanh c$$

In order to determine a parametrization of the geodesic $AB$ on $H^2$ we observe that a normal to the Euclidean plane $OAB$ is

$$(\cosh c, \cos\alpha\sinh c, \sin\alpha\sinh c) \times (\cosh c, \cos\alpha\sinh c, -\sin\alpha\sinh c)$$

$$= 2\sin\alpha\sinh c(-\cos\alpha\sinh c, \cosh c, 0)$$

Hence, if $(x_1, x_2, x_3)$ is any point on the geodesic $AB$, then

$$x_1\cos\alpha\sinh c = x_2\cosh c$$

or

$$x_2 = x_1 \cos\alpha \tanh c = x_1 \tanh b$$

Conversion to the hyperbolic polar coordinates of (1) yields

$$x_1 \tanh b = x_2 = \sinh\eta \cos\theta = \sqrt{x_1^2 - 1}\cos\theta$$

or

$$x_1^2 \cos^2\theta - \cos^2\theta = x_1^2 \tanh^2 b$$

It follows that the geodesic $AB$ has the parametrization

$$\begin{cases} x_1(\theta) = \cos\theta(\cos^2\theta - \tanh^2 b)^{-1/2} \\ x_2(\theta) = \cos\theta \tanh b(\cos^2\theta - \tanh^2 b)^{-1/2} \\ x_3(\theta) = \sin\theta \tanh b(\cos^2\theta - \tanh^2 b)^{-1/2} \end{cases}$$

or

$$\mathbf{x}(\theta) = \frac{(\cos\theta, \cos\theta\tanh b, \sin\theta\tanh b)}{\sqrt{\cos^2\theta - \tanh^2 b}}$$

In the calculations below, $\mathbf{x}(\theta)P$ denotes the hyperbolic distance from the point $\mathbf{x}(\theta)$ on $AmB$ to $P$, and $(x_1, x_2, x_3)$ denotes an arbitrary point in $\Delta PAB$ on $H^2$. Let (I, II, III) be the precentroid of $\Delta PAB$. Symmetry dictates that III $= 0$. Moreover,

$$\begin{aligned} \mathrm{I} &= \int_{-\alpha}^{\alpha}\int_0^{\mathbf{x}(\theta)P} \sinh\eta x_1 d\eta d\theta = \int_{-\alpha}^{\alpha}\int_0^{\mathbf{x}(\theta)P} \sinh\eta\cosh\eta d\eta d\theta \\ &= \int_{-\alpha}^{\alpha} \frac{1}{2}\Big(\cosh^2(\mathbf{x}(\theta)P) - \cosh^2 0\Big) = \frac{1}{2}\int_{-\alpha}^{\alpha} x_1^2(\theta) d\theta - \alpha \\ &= \int_0^{\alpha} \frac{\cos^2\theta d\theta}{\cos^2\theta - \tanh^2 b} - \alpha \\ &= \Big[\theta + \tanh^{-1}(\sinh b \tan\theta)\sinh b\Big]_0^{\alpha} = a \sinh b \end{aligned}$$

$$\begin{aligned} \mathrm{II} &= \int_{-\alpha}^{\alpha}\int_0^{\mathbf{x}(\theta)P} \sinh\eta x_2 d\eta d\theta = \int_{-\alpha}^{\alpha}\int_0^{\mathbf{x}(\theta)P} \sinh^2\eta\cos\theta d\eta d\theta \\ &= \frac{1}{2}\int_{-\alpha}^{\alpha}\cos\theta \int_0^{\mathbf{x}(\theta)P}(\cosh 2\eta - 1) d\eta d\theta \\ &= \frac{1}{4}\int_{-\alpha}^{\alpha}\cos\theta\left(\sinh 2\mathbf{x}(\theta)P - 2\mathbf{x}(\theta)P\right) d\theta \\ &= \frac{1}{4}\int_{-\alpha}^{\alpha}\cos\theta\left[2\cosh\mathbf{x}(\theta)P\sqrt{\cosh^2\mathbf{x}(\theta)P - 1} - 2\mathbf{x}(\theta)P\right] d\theta \\ &= \frac{1}{2}\int_{-\alpha}^{\alpha}\cos\theta\left[x_1(\theta)\sqrt{x_1^2(\theta) - 1} - \cosh^{-1} x_1(\theta)\right] d\theta \end{aligned}$$

$$= \tanh b \int_0^\alpha \frac{\cos^2\theta d\theta}{\cos^2\theta - \tanh^2 b}$$

$$- \int_0^\alpha \cos\theta \cosh^{-1}\left(\frac{\cos\theta}{\sqrt{\cos^2\theta - \tanh^2 b}}\right) d\theta$$

$$= (\alpha + a\sinh b)\tanh b - \int_0^\alpha \cos\theta \tanh^{-1}(\tanh b \sec\theta) d\theta$$

$$= (\alpha + a\sinh b)\tanh b$$

$$- \Big[\tanh^{-1}(\tanh b\sec\theta)\sin\theta - \tanh^{-1}(\sinh b\tan\theta)/\cosh b + \theta\tanh b\Big]_0^\alpha$$

$$= (\alpha + a\sinh b)\tanh b - (c\sin\alpha - a \text{ sech } b + \alpha\tanh b)$$

$$= a\cosh b - c\sin\alpha$$

These expressions for I and II together with Eqn's (3, 5) yield the following proposition.

**Proposition 1.2** *In a hyperbolic isosceles triangle with equal sides $c$ and vertex angle $2\alpha$ the distance from the vertex to the triangle's centroid is*

$$\tanh^{-1}\left(\frac{a\cosh b - c\sin\alpha}{a\sinh b}\right)$$

□

Let $C$ denote the centroid of the $\Delta ABC$ and let $M$ be the intersection of its medians. If $\alpha = \pi/4$ and $c = 1$ then

$$PC = 0.417455..., PM = 0.434114...$$

and so $C \neq M$. Thus, in contrast with the situation in Euclidean geometry, the centroid of a triangle and the center of mass of a uniform point-mass system located at its vertices are in general distinct. It would be interesting to find a (necessarily non-uniform) mass distribution on the vertices of a triangle whose center of mass agrees with its centroid.

The asymptotic behavior of $\delta(\alpha, c)$ is similar to that of $d(\alpha, c)$. Since

$$\frac{\delta(\alpha,c)}{b} = \frac{1}{b}\tanh^{-1}\left(\frac{a\cosh b - c\sin\alpha}{a\sinh b}\right) = \left(\frac{2}{3} + O(\alpha^3)\right) + O(c^3)$$

it follows that

$$\lim_{c\to 0}\frac{\delta(\alpha,c)}{PD} = \frac{2}{3}.$$

On the other hand,

$$\lim_{c\to\infty}\frac{\delta(\alpha,c)}{PD} = \lim_{c\to\infty}\frac{\tanh^{-1}\left(\coth b - \frac{c}{a}\frac{\sin\alpha}{\sinh b}\right)}{\tanh^{-1}(\cos\alpha\tanh b)}$$

$$= \frac{\tanh^{-1}\left(\sec\alpha - 1 \cdot \frac{\sin\alpha}{\cot\alpha}\right)}{\tanh^{-1}(\cos\alpha)} = 1$$

**Acknowledgements:** The author is indebted to his colleague David Lerner for his help and patience.

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